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A CLASS OF C^*-ALGEBRAS GENERALIZING BOTH GRAPH ALGEBRAS AND HOMEOMORPHISM C^*-ALGEBRAS III, IDEAL STRUCTURES

TAKESHI KATSURA

Abstract. We investigate the ideal structures of the C^*-algebras arising from topological graphs. We give the complete description of ideals of such C^*-algebras which are invariant under the so-called gauge action, and give the condition on topological graphs so that all ideals are invariant under the gauge action. We get conditions for our C^*-algebras to be simple, prime or primitive. We completely determine the prime ideals, and show that most of them are primitive. Finally, we construct a discrete graph such that the associated C^*-algebra is prime but not primitive.

0. Introduction

From a homeomorphism on some locally compact space, we can construct a C^*-algebra called a homeomorphism C^*-algebra (or a crossed product). The ideal structure of the homeomorphism C^*-algebra reflects the orbit structure of the given homeomorphism (see [Wi, T2, T3] for example). On the other hand, the ideal structures of graph algebras have been examined by many researchers (for example, [HR, KPRR, BPRS, BHRS, DT, HS]). These two lines of analysis have several similar aspects in common. Topological graphs introduced in [K1] generalize dynamical systems and (discrete) graphs, and the construction of C^*-algebras from topological graphs defined in [K1] generalizes the ones of homeomorphism C^*-algebras and graph algebras. In this paper, we unify the two analyses of ideal structures, and generalize them to the setting of topological graphs. The purposes of this paper include giving a dynamical insight into the theory of graphs and graph algebras. We mainly borrow terminologies from the theory of dynamical systems.

In Section 1, we recall the definition of topological graphs, and the way to construct a C^*-algebra $\mathcal{O}(E)$ from a topological graph E. In Section 2, we introduce the notion of invariant sets and admissible pairs of a topological graph E, and see that admissible pairs arise from ideals of the C^*-algebra $\mathcal{O}(E)$. Conversely, in Section 3, we see that an ideal of the C^*-algebra $\mathcal{O}(E)$ arises from each admissible pair of E. We show that by these correspondences, the set of all gauge-invariant ideals of $\mathcal{O}(E)$ corresponds bijectively to the set of all admissible pairs of E (Theorem 3.19). The key ingredient for the proof of this theorem is the observation done in [K2, Section 3]. In Section 4, we introduce the notion of orbits which generalizes the one for dynamical systems. In Section 5, we study hereditary and saturated sets which are generalizations of the ones in the theory of graph algebras. As an application, we give a characterization of maximal heads, which is defined in the previous section. In Section 6, we see that some of the ideals of $\mathcal{O}(E)$ are strongly Morita equivalent to the C^*-algebras of subgraphs of E. Using this observation, we show that the topological freeness is needed in the Cuntz-Krieger Uniqueness Theorem (Theorem 6.14). In Section 7, we define periodic and aperiodic points of a topological graph, and freeness of topological graphs. We show that a topological graph E is free if

and only if all ideals of $\mathcal{O}(E)$ are gauge-invariant (Theorem 7.6). In Sections 8 and 10, we generalize the notions of minimality and topological transitivity from dynamical systems to topological graphs, and give a couple of equivalent conditions on a topological graph E so that the C^*-algebra $\mathcal{O}(E)$ is simple (Theorem 8.12) and prime (Theorem 10.3), respectively. For the primeness of the C^*-algebra $\mathcal{O}(E)$, we use the results in Section 9 where we introduce the primeness for admissible pairs, and completely determine prime admissible pairs. Using the analysis in Section 9, we completely determine prime ideals of our C^*-algebras (Theorem 11.14). In Section 12, we give some ways to construct irreducible representations of our C^*-algebras, and show that most of their prime ideals are primitive (Theorem 12.1). In Section 13, we give one sufficient condition and one necessary condition on a topological graph E so that the C^*-algebra $\mathcal{O}(E)$ is primitive. Finally, we construct a discrete graph E such that $\mathcal{O}(E)$ is prime but not primitive (a similar construction can be found in [K4]). This C^*-algebra is an inductive limit of finite dimensional C^*-algebras and simultaneously residually finite dimensional. Thus $\mathcal{O}(E)$ is an easier example of a prime C^*-algebra which is not primitive than the first such example found in [We].

While this paper was under construction, P. S. Muhly and M. Tomforde introduced topological quivers in [MT] which include topological graphs as special examples. In the same paper, they give the method to construct C^*-algebras from them which generalizes our construction. Among others, they analyze the ideal structures of their C^*-algebras, and so some of our results are generalized to their general setting. In particular, Theorems 3.19, 6.14 and 8.12 in this paper are valid for topological quivers without significant changes of statements and proofs.

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1. Preliminaries

Definition 1.1. A *topological graph* $E = (E^0, E^1, d, r)$ consists of two locally compact spaces E^0 and E^1, and two maps $d, r\colon E^1 \to E^0$, where d is locally homeomorphic and r is continuous.

In this paper, $E = (E^0, E^1, d, r)$ always means a topological graph. The triple (E^1, d, r) is called a topological correspondence over E^0 in [K1], which can be considered as a multi-valued continuous map. A (topological) dynamical system is a pair $\Sigma = (X, \sigma)$ consisting of a locally compact space X and a homeomorphism σ on X. From a dynamical system $\Sigma = (X, \sigma)$, we get a topological graph E_Σ by $E_\Sigma = (X, X, \mathrm{id}_X, \sigma)$. We regard topological graphs as a generalization of dynamical systems and borrow many notions from the theory of dynamical systems. Sometimes, we think E^0 as a set of vertices and E^1 as a set of edges, and that an edge $e \in E^1$ is directed from its domain $d(e) \in E^0$ to its range $r(e) \in E^0$. This viewpoint explains why $E = (E^0, E^1, d, r)$ is called a topological graph.

Let us denote by $C_d(E^1)$ the set of continuous functions ξ on E^1 such that $\langle \xi, \xi \rangle(v) = \sum_{e \in d^{-1}(v)} |\xi(e)|^2 < \infty$ for any $v \in E^0$ and $\langle \xi, \xi \rangle \in C_0(E^0)$. For $\xi, \eta \in C_d(E^1)$ and

$f \in C_0(E^0)$, we define $\xi f \in C_d(E^1)$ and $\langle \xi, \eta \rangle \in C_0(E^0)$ by

$$(\xi f)(e) = \xi(e) f(d(e)) \text{ for } e \in E^1$$

$$\langle \xi, \eta \rangle(v) = \sum_{e \in d^{-1}(v)} \overline{\xi(e)} \eta(e) \text{ for } v \in E^0.$$

With these operations, $C_d(E^1)$ is a (right) Hilbert $C_0(E^0)$-module ([K1, Proposition 1.10]). We define a left action π_r of $C_0(E^0)$ on $C_d(E^1)$ by $(\pi_r(f)\xi)(e) = f(r(e))\xi(e)$ for $e \in E^1$, $\xi \in C_d(E^1)$ and $f \in C_0(E^0)$. Thus we get a C^*-correspondence $C_d(E^1)$ over $C_0(E^0)$.

We set $d^0 = r^0 = \mathrm{id}_{E^0}$ and $d^1 = d, r^1 = r$. For $n = 2, 3, \ldots$, we recursively define a space E^n of paths with length n and domain and range maps $d^n, r^n \colon E^n \to E^0$ by

$$E^n = \{(e', e) \in E^1 \times E^{n-1} \mid d^1(e') = r^{n-1}(e)\},$$

$d^n((e', e)) = d^{n-1}(e)$ and $r^n((e', e)) = r^1(e')$. We can define a C^*-correspondence $C_{d^n}(E^n)$ over $C_0(E^0)$ similarly as $C_d(E^1)$. We have $C_{d^{n+m}}(E^{n+m}) \cong C_{d^n}(E^n) \otimes C_{d^m}(E^m)$ as C^*-correspondences over $C_0(E^0)$ for any $n, m \in \mathbb{N} = \{0, 1, 2, \ldots\}$. As long as no confusion arises, we omit the superscript n and simply write d, r for d^n, r^n. Thus we get two maps $d, r \colon E^* \to E^0$ where $E^* = \coprod_{n=0}^{\infty} E^n$ is the *finite path space* of the topological graph E.

Definition 1.2. A *Toeplitz E-pair* on a C^*-algebra A is a pair of maps $T = (T^0, T^1)$ where $T^0 \colon C_0(E^0) \to A$ is a $*$-homomorphism and $T^1 \colon C_d(E^1) \to A$ is a linear map satisfying that

(i) $T^1(\xi)^* T^1(\eta) = T^0(\langle \xi, \eta \rangle)$ for $\xi, \eta \in C_d(E^1)$,
(ii) $T^0(f) T^1(\xi) = T^1(\pi_r(f)\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$.

For a Toeplitz E-pair $T = (T^0, T^1)$, the equation $T^1(\xi) T^0(f) = T^1(\xi f)$ holds automatically from the condition (i). We write $C^*(T)$ for denoting the C^*-algebra generated by the images of the maps T^0 and T^1. For $n \geq 2$, we can define a linear map $T^n \colon C_d(E^n) \to C^*(T)$ by $T^n(\xi) = T^1(\xi_1) T^1(\xi_2) \cdots T^1(\xi_n)$ for $\xi = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n \in C_d(E^n)$. The linear space

$$\operatorname{span}\{T^n(\xi) T^m(\eta)^* \mid \xi \in C_d(E^n),\ \eta \in C_d(E^m),\ n, m \in \mathbb{N}\}$$

is dense in $C^*(T)$ (see the remark after [K1, Lemma 2.4]).

We say that a Toeplitz E-pair $T = (T^0, T^1)$ is *injective* if T^0 is injective. If a Toeplitz E-pair $T = (T^0, T^1)$ is injective, then T^n are isometric for all $n \in \mathbb{N}$. We say a Toeplitz E-pair T *admits a gauge action* if there exists an automorphism β'_z on $C^*(T)$ with $\beta'_z(T^0(f)) = T^0(f)$ and $\beta'_z(T^1(\xi)) = z T^1(\xi)$ for every $z \in \mathbb{T}$. If T admits a gauge action β', then we have

$$\beta'_z(T^n(\xi) T^m(\eta)^*) = z^{n-m} T^n(\xi) T^m(\eta)^*$$

for $\xi \in C_d(E^n)$ and $\eta \in C_d(E^m)$.

Definition 1.3. We define three open subsets $E^0_{\mathrm{sce}}, E^0_{\mathrm{fin}}$ and E^0_{rg} of E^0 by $E^0_{\mathrm{sce}} = E^0 \setminus \overline{r(E^1)}$,

$$E^0_{\mathrm{fin}} = \{v \in E^0 \mid \text{ there exists a neighborhood } V \text{ of } v \\ \text{such that } r^{-1}(V) \subset E^1 \text{ is compact}\},$$

and $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$. We define two closed subsets E^0_{inf} and E^0_{sg} of E^0 by $E^0_{\mathrm{inf}} = E^0 \setminus E^0_{\mathrm{fin}}$ and $E^0_{\mathrm{sg}} = E^0 \setminus E^0_{\mathrm{rg}}$.

A vertex in $E^0_{\rm sce}$ is called a *source*. We have $E^0_{\rm sg} = E^0_{\rm inf} \cup \overline{E^0_{\rm sce}}$. Vertices in $E^0_{\rm rg}$ are said to be *regular*, and those in $E^0_{\rm sg}$ are said to be *singular*. We have that $\pi_r^{-1}(\mathcal{K}(C_d(E^1))) = C_0(E^0_{\rm fin})$ and $\ker \pi_r = C_0(E^0_{\rm sce})$ ([K1, Proposition 1.24]). Hence the restriction of π_r to $C_0(E^0_{\rm rg})$ is an injection into $\mathcal{K}(C_d(E^1))$. When a topological graph E is defined from a dynamical system Σ, we get $E^0_{\rm rg} = E^0$. The set of regular vertices $E^0_{\rm rg}$ is the maximal open subset such that the restriction $r\colon r^{-1}(E^0_{\rm rg}) \to E^0_{\rm rg}$ is surjective and proper. This fact implies the following property of regular vertices, which roughly says that $E^0_{\rm rg}$ is the part in which the topological correspondence (E^1, d, r) is "reversible".

Lemma 1.4 ([K1, Lemma 1.21])**.** *For $v \in E^0_{\rm rg}$, the set $r^{-1}(v) \subset E^1$ is a non-empty compact set, and for an open subset U of E^1 with $r^{-1}(v) \subset U$ there exists a neighborhood V of v such that $r^{-1}(V) \subset U$.*

For a Toeplitz E-pair $T = (T^0, T^1)$, we define a $*$-homomorphism $\Phi\colon \mathcal{K}(C_d(E^1)) \to C^*(T)$ by $\Phi(\theta_{\xi,\eta}) = T^1(\xi)T^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$.

Definition 1.5. A Toeplitz E-pair $T = (T^0, T^1)$ is called a *Cuntz-Krieger E-pair* if $T^0(f) = \Phi(\pi_r(f))$ for any $f \in C_0(E^0_{\rm rg})$.

The C^*-algebra $\mathcal{O}(E)$ is generated by the universal Cuntz-Krieger E-pair $t = (t^0, t^1)$.

Since t^0 is injective ([K1, Proposition 3.7]), the $*$-homomorphism $\varphi\colon \mathcal{K}(C_d(E^1)) \to \mathcal{O}(E)$ is injective, and the maps $t^n\colon C_d(E^n) \to \mathcal{O}(E)$ are isometric for all $n \in \mathbb{N}$. The universal Cuntz-Krieger E-pair $t = (t^0, t^1)$ admits a gauge action, which will be denoted by $\beta\colon \mathbb{T} \curvearrowright \mathcal{O}(E)$.

2. Admissible pairs

In this section, we introduce invariant sets and admissible pairs for a topological graph E, and see that admissible pairs correspond to ideals of the C^*-algebra $\mathcal{O}(E)$.

Definition 2.1. A subset X^0 of E^0 is said to be *positively invariant* if $d(e) \in X^0$ implies $r(e) \in X^0$ for each $e \in E^1$, and to be *negatively invariant* if for $v \in X^0 \cap E^0_{\rm rg}$, there exists $e \in E^1$ with $r(e) = v$ and $d(e) \in X^0$. A subset X^0 of E^0 is said to be *invariant* if X^0 is both positively and negatively invariant.

These terminologies come from regarding $E = (E^0, E^1, d, r)$ as a generalization of dynamical systems.

For a closed positively invariant subset X^0, we define a closed subset X^1 of E^1 by $X^1 = d^{-1}(X^0)$. Since X^0 is positively invariant, we have $r(X^1) \subset X^0$. Hence $X = (X^0, X^1, d_X, r_X)$ is a topological graph, where d_X and r_X are the restrictions of d and r to X^1.

Proposition 2.2. *For a closed positively invariant subset X^0 of E^0, the following conditions are equivalent;*

(i) X^0 *is negatively invariant,*
(ii) $X^0_{\rm sce} \cap E^0_{\rm rg} = \emptyset$,
(iii) $X^0_{\rm sg} \subset E^0_{\rm sg}$.

Proof. (i)⇒(ii): If X^0 is negatively invariant, then every $v \in X^0 \cap E^0_{\rm rg}$ satisfies that $r_X^{-1}(v) = r^{-1}(v) \cap X^1 \neq \emptyset$. Hence $X^0_{\rm sce} \cap E^0_{\rm rg} = \emptyset$.

(ii)⇒(iii): From (ii) we have $X^0_{\mathrm{sce}} \subset E^0_{\mathrm{sg}}$. Thus $\overline{X^0_{\mathrm{sce}}} \subset E^0_{\mathrm{sg}}$. Clearly $X^0_{\mathrm{inf}} \subset E^0_{\mathrm{inf}} \subset E^0_{\mathrm{sg}}$. Therefore we have $X^0_{\mathrm{sg}} \subset E^0_{\mathrm{sg}}$.

(iii)⇒(i): Take $v \in X^0 \cap E^0_{\mathrm{rg}}$. From (iii), we have $v \in X^0_{\mathrm{rg}}$. Hence there exists $e \in X^1 = d^{-1}(X^0)$ with $r(e) = v$ by Lemma 1.4. Thus X^0 is negatively invariant. □

By Proposition 2.2, we have an inclusion $X^0_{\mathrm{sg}} \subset E^0_{\mathrm{sg}} \cap X^0$ for a closed invariant set X^0. There are many examples in which this inclusion is proper (see Example 3.21 and Proposition 3.24).

Definition 2.3. A pair $\rho = (X^0, Z)$ consisting of two closed subsets X^0 and Z of E^0 is called an *admissible pair* if X^0 is invariant and $X^0_{\mathrm{sg}} \subset Z \subset E^0_{\mathrm{sg}} \cap X^0$.

Admissible pairs naturally arise from ideals of $\mathcal{O}(E)$. We denote by $\mathcal{G}^0$ and $\mathcal{G}^1$ the images of $t^0\colon C_0(E^0) \to \mathcal{O}(E)$ and $\varphi\colon \mathcal{K}(C_d(E^1)) \to \mathcal{O}(E)$, respectively.

Definition 2.4. For an ideal I of $\mathcal{O}(E)$, we define closed subsets X^0_I, Z_I of E^0 by

$$X^0_I = \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I\},$$
$$Z_I = \{v \in E^0 \mid f(v) = 0 \text{ for all } f \in C_0(E^0) \text{ with } t^0(f) \in I + \mathcal{G}^1\}.$$

The closed sets $X^0_I, Z_I \subset E^0$ are determined by

$$t^0(C_0(E^0 \setminus X^0_I)) = I \cap \mathcal{G}^0, \quad t^0(C_0(E^0 \setminus Z_I)) = (I + \mathcal{G}^1) \cap \mathcal{G}^0.$$

We denote by ρ_I the pair (X^0_I, Z_I) of closed subsets of E^0. We will show that the pair $\rho_I = (X^0_I, Z_I)$ is admissible.

Proposition 2.5. *For an ideal I of $\mathcal{O}(E)$, the closed set X^0_I is positively invariant.*

Proof. Take $e \in E^1$ with $d(e) \in X^0_I$. Take $f \in C_0(E^0)$ with $t^0(f) \in I$ arbitrarily, and we will show that $f(r(e)) = 0$. There exists $\xi \in C_d(E^1)$ with $\xi(e) = 1$ and $\xi(e') = 0$ for all $e' \in d^{-1}(d(e)) \setminus \{e\}$ because e is isolated in $d^{-1}(d(e))$. We have

$$\langle \xi, \pi_r(f)\xi\rangle(d(e)) = \sum_{e' \in d^{-1}(d(e))} \overline{\xi(e')} f(r(e'))\xi(e') = f(r(e)).$$

From $t^0(\langle \xi, \pi_r(f)\xi\rangle) = t^1(\xi)^* t^0(f) t^1(\xi) \in I$ and $d(e) \in X^0_I$, we get $\langle \xi, \pi_r(f)\xi\rangle(d(e)) = 0$. Thus we have $f(r(e)) = 0$. Hence $r(e) \in X^0_I$. Therefore X^0_I is positively invariant. □

Lemma 2.6. *Let I be an ideal of $\mathcal{O}(E)$.*

(i) *For $\xi \in C_d(E^1)$, $t^1(\xi) \in I$ if and only if $\xi \in C_d(E^1 \setminus X^1_I)$.*

(ii) *For $x \in \mathcal{K}(C_d(E^1))$, $\varphi(x) \in I$ if and only if $x\xi \in C_d(E^1 \setminus X^1_I)$ for all $\xi \in C_d(E^1)$.*

Proof. (i) For $\xi \in C_d(E^1)$, $t^1(\xi) \in I$ if and only if $t^1(\xi)^* t^1(\xi) = t^0(\langle \xi, \xi\rangle)$ is in I, which is equivalent to $\langle \xi, \xi\rangle \in C_0(E^0 \setminus X^0_I)$ by the definition of X^0_I. By [K1, Lemma 1.12], $\langle \xi, \xi\rangle \in C_0(E^0 \setminus X^0_I)$ is equivalent to $\xi \in C_d(E^1 \setminus X^1_I)$.

(ii) Take $x \in \mathcal{K}(C_d(E^1))$ with $\varphi(x) \in I$. For $\xi \in C_d(E^1)$, we have $x\xi \in C_d(E^1 \setminus X^1_I)$ by (i) because $t^1(x\xi) = \varphi(x)t^1(\xi) \in I$. Conversely suppose that $x \in \mathcal{K}(C_d(E^1))$ satisfies $x\xi \in C_d(E^1 \setminus X^1_I)$ for all $\xi \in C_d(E^1)$. Then x is approximated by elements in the form $\sum_{k=1}^{K} \theta_{\xi_k,\eta_k}$ where $\xi_k, \eta_k \in C_d(E^1 \setminus X^1_I)$ by [K1, Lemma 1.13]. Since we have

$$\varphi\Big(\sum_{k=1}^{K} \theta_{\xi_k,\eta_k}\Big) = \sum_{k=1}^{K} t^1(\xi_k) t^1(\eta_k)^* \in I$$

by (i), we get $\varphi(x) \in I$.

□

Proposition 2.7. *For an ideal I of $\mathcal{O}(E)$, the closed set X_I^0 is negatively invariant.*

Proof. Take $v \in E^0_{\mathrm{rg}}$ such that $d(e) \notin X_I^0$ for all $e \in r^{-1}(v)$, and we will prove that $v \notin X_I^0$. By Lemma 1.4, we can find a neighborhood $V \subset E^0_{\mathrm{rg}}$ of v so that $r^{-1}(V) \cap X_I^1 = \emptyset$. Take $f \in C_0(V)$ with $f(v) = 1$. For $\xi \in C_d(E^1)$ and $e \in X_I^1$, we have $(\pi_r(f)\xi)(e) = f(r(e))\xi(e) = 0$. Hence we see that $\pi_r(f)\xi \in C_d(E^1 \setminus X_I^1)$ for all $\xi \in C_d(E^1)$. By Lemma 2.6 (ii), we have $\varphi(\pi_r(f)) \in I$. Since $f \in C_0(E^0_{\mathrm{rg}})$, we have $t^0(f) = \varphi(\pi_r(f)) \in I$. Therefore $v \notin X_I^0$. This shows that X_I^0 is negatively invariant. □

Proposition 2.8. *For an ideal I of $\mathcal{O}(E)$, the pair $\rho_I = (X_I^0, Z_I)$ is admissible.*

Proof. We have already seen that X_I^0 is invariant in Proposition 2.5 and Proposition 2.7. Since $t^0(C_0(E^0_{\mathrm{rg}})) \subset \mathcal{G}^1$, we have $Z_I \cap E^0_{\mathrm{rg}} = \emptyset$. We also have $Z_I \subset X_I^0$ because $t^0(C_0(E^0 \setminus X_I^0)) \subset I$. Hence we get $Z_I \subset E^0_{\mathrm{sg}} \cap X_I^0$. We will show that $(X_I^0)_{\mathrm{sg}} \subset Z_I$. To derive a contradiction, suppose that there exists $v_0 \in (X_I^0)_{\mathrm{sg}}$ with $v_0 \notin Z_I$. Then there exist $f \in C_0(E^0)$ with $t^0(f) \in I + \mathcal{G}^1$ and $f(v_0) \neq 0$. Take $x \in \mathcal{K}(C_d(E^1))$ such that $t^0(f) + \varphi(x) \in I$. Since $v_0 \in (X_I^0)_{\mathrm{sg}}$, either $v_0 \in \overline{(X_I^0)_{\mathrm{sce}}}$ or $v_0 \in (X_I^0)_{\mathrm{inf}}$.

First we consider the case that $v_0 \in \overline{(X_I^0)_{\mathrm{sce}}}$. We can find $v_1 \in (X_I^0)_{\mathrm{sce}}$ with $f(v_1) \neq 0$. Since $v_1 \notin \overline{r(X_I^1)}$, there exists $g \in C_0(E^0)$ with $g(v_1) \neq 0$ and $g(v) = 0$ for $v \in r(X_I^1)$. For $\xi \in C_d(E^1)$ and $e \in X_I^1$, we have $(\pi_r(g)x\xi)(e) = g(r(e))\,(x\xi)(e) = 0$. Hence $(\pi_r(g)x)\xi \in C_d(E^1 \setminus X_I^1)$ for all $\xi \in C_d(E^1)$. By Lemma 2.6 (ii), we have $\varphi(\pi_r(g)x) \in I$. Since we have

$$t^0(gf) + \varphi(\pi_r(g)x) = t^0(g)\big(t^0(f) + \varphi(x)\big) \in I,$$

we get $t^0(gf) \in I$. This contradicts the fact that $(gf)(v_1) \neq 0$ and $v_1 \in X_I^0$.

Next we consider the case that $v_0 \in (X_I^0)_{\mathrm{inf}}$. We can find $\varepsilon > 0$ and a neighborhood V of v_0 such that $|f(v)| \geq \varepsilon$ for all $v \in V$. There exists $x' \in \mathcal{K}(C_d(E^1))$ satisfying that $\|x - x'\| < \varepsilon$ and $x' = \sum_{i=1}^k \theta_{\xi_i,\eta_i}$ for some $\xi_i, \eta_i \in C_c(E^1)$. We set $K = \bigcup_{i=1}^k \operatorname{supp} \eta_i$, which is a compact subset of E^1. Since $v_0 \in (X_I^0)_{\mathrm{inf}}$, we have $r^{-1}(V) \cap X_I^1 \not\subset K$. Take $e \in (r^{-1}(V) \cap X_I^1) \setminus K$. We can find $\xi \in C_d(E^1)$ such that $\|\xi\| = 1$, $\xi(e) = 1$ and $K \cap \operatorname{supp} \xi = \emptyset$. Since $t^1(\pi_r(f)\xi + x\xi) = \big(t^0(f) + \varphi(x)\big)t^1(\xi) \in I$, we have $(\pi_r(f)\xi + x\xi)(e) = 0$ by Lemma 2.6 (i). Since $\|x\xi\| = \|(x - x')\xi\| < \varepsilon$, we have $|x\xi(e)| < \varepsilon$. On the other hand, we have

$$|(\pi_r(f)\xi)(e)| = |f(r(e))\xi(e)| \geq \varepsilon.$$

This is a contradiction. The proof is completed. □

For two admissible pairs $\rho_1 = (X_1^0, Z_1)$, $\rho_2 = (X_2^0, Z_2)$, we write $\rho_1 \subset \rho_2$ if $X_1^0 \subset X_2^0$ and $Z_1 \subset Z_2$. By the definition, we can see the following.

Lemma 2.9. *For two ideals I_1, I_2 of $\mathcal{O}(E)$, $I_1 \subset I_2$ implies $\rho_{I_1} \supset \rho_{I_2}$.*

For two admissible pairs $\rho_1 = (X_1^0, Z_1)$, $\rho_2 = (X_2^0, Z_2)$, we denote by $\rho_1 \cup \rho_2$ the pair $(X_1^0 \cup X_2^0, Z_1 \cup Z_2)$. We will show that the pair $\rho_1 \cup \rho_2$ is admissible.

Lemma 2.10. *For two invariant closed sets X_1^0, X_2^0, the closed set $X^0 = X_1^0 \cup X_2^0$ is invariant, and we have $X^0_{\mathrm{sce}} \subset (X_1^0)_{\mathrm{sce}} \cup (X_2^0)_{\mathrm{sce}}$, $X^0_{\mathrm{inf}} = (X_1^0)_{\mathrm{inf}} \cup (X_2^0)_{\mathrm{inf}}$ and $X^0_{\mathrm{sg}} \subset (X_1^0)_{\mathrm{sg}} \cup (X_2^0)_{\mathrm{sg}}$.*

Proof. Take $e \in E^1$ with $d(e) \in X^0$. Then either $d(e) \in X_1^0$ or $d(e) \in X_2^0$. Since X_1^0 and X_2^0 are positively invariant, either $r(e) \in X_1^0$ or $r(e) \in X_2^0$ holds. Hence $r(e) \in X^0$. Thus X^0 is positively invariant.

Next we will show that $X^0_{\mathrm{sce}} \subset (X_1^0)_{\mathrm{sce}} \cup (X_2^0)_{\mathrm{sce}}$. Take $v \in X^0_{\mathrm{sce}}$. There exists a neighborhood V of $v \in X^0$ such that $r^{-1}(V) \cap X^1 = \emptyset$. When $v \in X_1^0$, the set $V \cap X_1^0$ is a neighborhood of $v \in X_1^0$ and we have $r^{-1}(V \cap X_1^0) \cap X_1^1 = \emptyset$ because $X_1^1 \subset X^1$. Hence we have $v \in (X_1^0)_{\mathrm{sce}}$. Similarly we get $v \in (X_2^0)_{\mathrm{sce}}$ when $v \in X_2^0$. Thus we have shown that $X^0_{\mathrm{sce}} \subset (X_1^0)_{\mathrm{sce}} \cup (X_2^0)_{\mathrm{sce}}$. Hence we have

$$X^0_{\mathrm{sce}} \cap E^0_{\mathrm{rg}} \subset \big((X_1^0)_{\mathrm{sce}} \cap E^0_{\mathrm{rg}}\big) \cup \big((X_2^0)_{\mathrm{sce}} \cap E^0_{\mathrm{rg}}\big) = \emptyset$$

because X_1^0 and X_2^0 are negatively invariant. By Proposition 2.2, X^0 is negatively invariant.

Next we will prove $X^0_{\mathrm{inf}} = (X_1^0)_{\mathrm{inf}} \cup (X_2^0)_{\mathrm{inf}}$. Take $v \in (X_1^0)_{\mathrm{inf}}$. For any closed neighborhood V of $v \in X^0$, $V \cap X_1^0$ is a closed neighborhood of $v \in X_1^0$. Since $v \in (X_1^0)_{\mathrm{inf}}$, we have that $r^{-1}(V \cap X_1^0) \cap X_1^1$ is not compact. From $r^{-1}(V) \cap X^1 \supset r^{-1}(V \cap X_1^0) \cap X_1^1$, we see that $r^{-1}(V) \cap X^1$ is not compact. Hence $v \in X^0_{\mathrm{inf}}$. Similarly if $v \in (X_2^0)_{\mathrm{inf}}$ then $v \in X^0_{\mathrm{inf}}$. Thus we have shown that $v \in X^0_{\mathrm{inf}}$ for every $v \in (X_1^0)_{\mathrm{inf}} \cup (X_2^0)_{\mathrm{inf}}$. Conversely take $v \notin (X_1^0)_{\mathrm{inf}} \cup (X_2^0)_{\mathrm{inf}}$, and we will show that $v \notin X^0_{\mathrm{inf}}$. From $v \notin (X_1^0)_{\mathrm{inf}}$, we have either $v \notin X_1^0$ or $v \in (X_1^0)_{\mathrm{fin}}$. For both cases, we can find a compact neighborhood V_1 of $v \in E^0$ such that $r^{-1}(V_1 \cap X_1^0) \cap X_1^1$ is compact. Similarly there exists a compact neighborhood V_2 of $v \in E^0$ such that $r^{-1}(V_2 \cap X_2^0) \cap X_2^1$ is compact. Set $V = V_1 \cap V_2$ which is a compact neighborhood of $v \in E^0$. We have

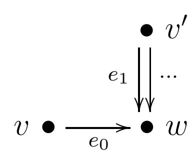
$$\begin{aligned} r^{-1}(V) \cap X^1 &= (r^{-1}(V) \cap X_1^1) \cup (r^{-1}(V) \cap X_2^1) \\ &= (r^{-1}(V \cap X_1^0) \cap X_1^1) \cup (r^{-1}(V \cap X_2^0) \cap X_2^1) \\ &\subset (r^{-1}(V_1 \cap X_1^0) \cap X_1^1) \cup (r^{-1}(V_2 \cap X_2^0) \cap X_2^1). \end{aligned}$$

Hence $r^{-1}(V) \cap X^1$ is compact. Thus $v \notin X^0_{\mathrm{inf}}$. Therefore we have $X^0_{\mathrm{inf}} = (X_1^0)_{\mathrm{inf}} \cup (X_2^0)_{\mathrm{inf}}$. Now it is easy to see that $X^0_{\mathrm{sg}} \subset (X_1^0)_{\mathrm{sg}} \cup (X_2^0)_{\mathrm{sg}}$. □

Note that in general $X_1^0 \subset X_2^0$ does not imply $(X_1^0)_{\mathrm{sg}} \subset (X_2^0)_{\mathrm{sg}}$ for two invariant closed sets X_1^0, X_2^0.

Example 2.11. Let $E = (E^0, E^1, d, r)$ be the discrete graph given by

$$E^0 = \{v, v', w\}, \qquad E^1 = \{e_k\}_{k \in \mathbb{N}},$$
$$d(e_k) = \begin{cases} v & (k = 0) \\ v' & (k \geq 1) \end{cases}, \quad r(e_k) = w \ (k \in \mathbb{N}).$$



This example is the same as in [K2, Example 4.9]. Since $E^0_{\mathrm{sg}} = E^0$, every subset of E^0 is negatively invariant. The two sets $X_1^0 = \{w\}$ and $X_2^0 = \{v, w\}$ are closed invariant sets satisfying $X_1^0 \subset X_2^0$. However we do not have $(X_1^0)_{\mathrm{sg}} \subset (X_2^0)_{\mathrm{sg}}$ because $(X_1^0)_{\mathrm{sg}} = \{w\}$ and $(X_2^0)_{\mathrm{sg}} = \{v\}$.

Proposition 2.12. *For two admissible pairs $\rho_1 = (X_1^0, Z_1)$, $\rho_2 = (X_2^0, Z_2)$, the pair $\rho_1 \cup \rho_2$ is admissible.*

Proof. By Lemma 2.10, $X^0 = X_1^0 \cup X_2^0$ is invariant and

$$X^0_{\mathrm{sg}} \subset (X_1^0)_{\mathrm{sg}} \cup (X_2^0)_{\mathrm{sg}} \subset Z_1 \cup Z_2 \subset (E^0_{\mathrm{sg}} \cap X_1^0) \cup (E^0_{\mathrm{sg}} \cap X_2^0) = E^0_{\mathrm{sg}} \cap X^0.$$

Therefore $\rho_1 \cup \rho_2$ is admissible. $\square$

Proposition 2.13. *For two ideals I_1, I_2 of $\mathcal{O}(E)$, we have $\rho_{I_1\cap I_2} = \rho_{I_1} \cup \rho_{I_2}$.*

Proof. This follows from the following computations

$$\begin{aligned} I_1 \cap I_2 \cap \mathcal{G}^0 &= (I_1 \cap \mathcal{G}^0) \cap (I_2 \cap \mathcal{G}^0) \\ \big(I_1 \cap I_2 + \mathcal{G}^1\big) \cap \mathcal{G}^0 &= \big((I_1 + \mathcal{G}^1) \cap (I_2 + \mathcal{G}^1)\big) \cap \mathcal{G}^0 \\ &= \big((I_1 + \mathcal{G}^1) \cap \mathcal{G}^0\big) \cap \big((I_2 + \mathcal{G}^1) \cap \mathcal{G}^0\big) \end{aligned}$$

and the remark after Definition 2.4. $\square$

3. Gauge-invariant ideals

In Section 2, we get admissible pairs ρ_I from ideals I of $\mathcal{O}(E)$. Conversely we can construct ideals I_ρ from admissible pairs ρ.

We set $\mathcal{F}^1 = \mathcal{G}^0 + \mathcal{G}^1 \subset \mathcal{O}(E)$ which is a C^*-subalgebra. We recall the description of $\mathcal{F}^1$ done in [K1, Proposition 5.2]. Two $*$-homomorphisms $\pi_0^1\colon \mathcal{F}^1 \to C_0(E^0_{\mathrm{sg}})$ and $\pi_1^1\colon \mathcal{F}^1 \to \mathcal{L}(C_d(E^1))$ are defined by $\pi_0^1(a) = f|_{E^0_{\mathrm{sg}}}$ and $\pi_1^1(a) = \pi_r(f) + x$ for $a = t^0(f) + \varphi(x) \in \mathcal{F}^1$. Note that π_0^1 is a surjective map whose kernel is $\mathcal{G}^1$, and that the restriction of π_1^1 to $\mathcal{G}^1$ is the inverse map of φ. Thus

$$\pi_0^1 \oplus \pi_1^1\colon \mathcal{F}^1 \to C_0(E^0_{\mathrm{sg}}) \oplus \mathcal{L}(C_d(E^1))$$

is injective. Note that we have $at^1(\xi) = t^1(\pi_1^1(a)\xi)$ for $a \in \mathcal{F}^1$ and $\xi \in C_d(E^1)$, and that for an ideal I of $\mathcal{O}(E)$ the closed set Z_I is determined by $C_0(E^0_{\mathrm{sg}} \setminus Z_I) = \pi_0^1(I \cap \mathcal{F}^1)$.

Let X^0 be a closed positively invariant subset of E^0. We get the topological graph $X = (X^0, X^1, d_X, r_X)$ where $X^1 = d^{-1}(X^0) \subset E^1$ and d_X, r_X are the restrictions of d, r to X^1. There is a natural surjection $C_d(E^1) \ni \xi \mapsto \dot{\xi} \in C_{d_X}(X^1)$ whose kernel is $C_d(E^1 \setminus X^1)$ ([K1, Lemma 1.12]). Since the submodule $C_d(E^1 \setminus X^1)$ is invariant under the action of $\mathcal{L}(C_d(E^1))$, we can define a $*$-homomorphism $\omega_X\colon \mathcal{L}(C_d(E^1)) \to \mathcal{L}(C_{d_X}(X^1))$ by $\omega_X(x)\dot{\xi} = (x\xi)^{\cdot}$. For $f \in C_0(E^0)$, we have $\omega_X(\pi_r(f)) = \pi_{r_X}(f|_{X^0})$. One can easily see that

$$\ker \omega_X = \{x \in \mathcal{L}(C_d(E^1)) \mid x\xi \in C_d(E^1 \setminus X^1) \text{ for all } \xi \in C_d(E^1)\}.$$

It is easy to see that the restriction of ω_X to $\mathcal{K}(C_d(E^1))$ is a surjection onto $\mathcal{K}(C_{d_X}(X^1))$ ([K1, Lemma 1.14]).

Definition 3.1. For an admissible pair $\rho = (X^0, Z)$, we define an ideal J_ρ of $\mathcal{F}^1$ by

$$\begin{aligned} J_\rho &= \{a \in \mathcal{F}^1 \mid \pi_0^1(a) \in C_0(E^0_{\mathrm{sg}} \setminus Z),\ \pi_1^1(a) \in \ker \omega_X\} \\ &= \{t^0(f) + \varphi(x) \in \mathcal{F}^1 \mid f \in C_0(E^0 \setminus Z),\ \omega_X(\pi_r(f) + x) = 0\}. \end{aligned}$$

Lemma 3.2. *The ideal J_ρ of $\mathcal{F}^1$ satisfies*

$$\pi_0^1(J_\rho) = C_0(E^0_{\mathrm{sg}} \setminus Z), \text{ and } J_\rho \cap \mathcal{G}^0 = t^0(C_0(E^0 \setminus X^0)).$$

Proof. By definition, $\pi_0^1(J_\rho) \subset C_0(E^0_{\mathrm{sg}} \setminus Z)$. We will show the other inclusion $\pi_0^1(J_\rho) \supset C_0(E^0_{\mathrm{sg}} \setminus Z)$. Take $g \in C_0(E^0_{\mathrm{sg}} \setminus Z)$. Choose $f \in C_0(E^0 \setminus Z)$ such that $f|_{E^0_{\mathrm{sg}}} = g$. Since $X^0_{\mathrm{sg}} \subset Z$, we have $f|_{X^0} \in C_0(X^0_{\mathrm{rg}})$. Thus we see $\omega_X(\pi_r(f)) = \pi_{r_X}(f|_{X^0}) \in \mathcal{K}(C_{d_X}(X^1))$. Hence there exists $x \in \mathcal{K}(C_d(E^1))$ such that $\omega_X(x) = \omega_X(\pi_r(f))$. Thus we get $a = t^0(f) - \varphi(x) \in J_\rho$ with $\pi_0^1(a) = g$. Therefore $\pi_0^1(J_\rho) = C_0(E^0_{\mathrm{sg}} \setminus Z)$.

For $f \in C_0(E^0)$, $t^0(f) \in J_\rho$ if and only if $f \in C_0(E^0 \setminus Z)$ and $\pi_r(f) \in \ker \omega_X$. Now $\pi_r(f) \in \ker \omega_X$ if and only if $f(v) = 0$ for all $v \in r(X^1)$. Since $X^0_{\mathrm{sg}} \subset Z \subset X^0$ and $X^0_{\mathrm{rg}} \subset r(X^1) \subset X^0$, we have $Z \cup r(X^1) = X^0$. Therefore, $t^0(f) \in J_\rho$ if and only if $f \in C_0(E^0 \setminus X^0)$. □

Definition 3.3. For an admissible pair $\rho = (X^0, Z)$, we define a subset I_ρ of $\mathcal{O}(E)$ to be the closure of

$$\operatorname{span}\{t^n(\xi) a t^m(\eta)^* \mid a \in J_\rho,\ \xi \in C_d(E^n),\ \eta \in C_d(E^m),\ n, m \in \mathbb{N}\}.$$

We will see that the set I_ρ is the ideal generated by J_ρ. To prove it, we need the following lemma.

Lemma 3.4. *For $a \in J_\rho$ and $\zeta \in C_d(E^k)$ with $k \geq 1$, there exist $\zeta' \in C_d(E^k)$ and $f \in C_0(E^0 \setminus X^0)$ such that $a t^k(\zeta) = t^k(\zeta') t^0(f)$.*

Proof. Let us set $X^k = (d^k)^{-1}(X^0) \subset E^k$. For $\zeta'' \in C_d(E^k \setminus X^k)$, we can find $\zeta' \in C_d(E^k)$ and $f \in C_0(E^0 \setminus X^0)$ with $t^k(\zeta'') = t^k(\zeta') t^0(f)$ by [K1, Lemma 1.12]. Hence it suffices to show $a t^k(\zeta) \in t^k(C_d(E^k \setminus X^k))$. We may assume $\zeta = \xi \otimes \eta$ for $\xi \in C_d(E^1)$ and $\eta \in C_d(E^{k-1})$. We have

$$a t^k(\zeta) = t^k((\pi^1_1(a)\xi) \otimes \eta).$$

Since $\pi^1_1(a) \in \ker \omega_X$, we have $\pi^1_1(a)\xi \in C_d(E^1 \setminus X^1)$. Since X^0 is positively invariant, $(e_1, \ldots, e_k) \in X^k$ implies $e_1 \in X^1$. Hence we get $(\pi^1_1(a)\xi) \otimes \eta \in C_d(E^k \setminus X^k)$. We are done. □

Proposition 3.5. *The set I_ρ is a gauge-invariant ideal of $\mathcal{O}(E)$.*

Proof. By definition, I_ρ is a closed $*$-invariant subspace of $\mathcal{O}(E)$, which is invariant under the gauge action. To prove that I_ρ is an ideal, it suffices to see that $xy \in I_\rho$ for $x = t^n(\xi) a t^m(\eta)^* \in I_\rho$ and $y = t^{n'}(\xi') t^{m'}(\eta')^* \in \mathcal{O}(E)$ where $a \in J_\rho$. When $m \geq n'$, obviously $xy \in I_\rho$. Hence we only need to show that $t^n(\xi) a t^k(\zeta) t^{m'}(\eta')^* \in I_\rho$ for some $k \geq 1$ and $\zeta \in C_d(E^k)$. By Lemma 3.4, there exist $\zeta' \in C_d(E^k)$ and $f \in C_0(E^0 \setminus X^0)$ such that $a t^k(\zeta) = t^k(\zeta') t^0(f)$. Thus we get

$$t^n(\xi) a t^k(\zeta) t^{m'}(\eta')^* = t^{n+k}(\xi \otimes \zeta') t^0(f) t^{m'}(\eta')^* \in I_\rho,$$

because $t^0(f) \in J_\rho$ by Lemma 3.2. This completes the proof. □

We will show that for an admissible pair ρ, we have $\rho_{I_\rho} = \rho$. To this end, we need a series of lemmas.

Lemma 3.6. *There exists a norm-decreasing map $\pi_0 \colon \mathcal{O}(E) \to C_0(E^0_{\mathrm{sg}})$ satisfying that $\pi_0(t^0(f)) = f|_{E^0_{\mathrm{sg}}}$ for $f \in C_0(E^0)$ and $\pi_0(t^n(\xi) t^m(\eta)^*) = 0$ when $n \geq 1$ or $m \geq 1$.*

Proof. Let $\mathcal{F} \subset \mathcal{O}(E)$ be the fixed point algebra of the gauge action β, and $\Psi \colon \mathcal{O}(E) \to \mathcal{F}$ be the conditional expectation defined by $\Psi(x) = \int_{\mathbb{T}} \beta_z(x) dz$ for $x \in \mathcal{O}(E)$. For $\xi \in C_d(E^n)$ and $\eta \in C_d(E^m)$, we have

$$\Psi(t^n(\xi) t^m(\eta)^*) = \delta_{n,m} t^n(\xi) t^m(\eta)^*.$$

We can define a $*$-homomorphism $\pi'_0 \colon \mathcal{F} \to C_0(E^0_{\mathrm{sg}})$ by $\pi'_0(t^0(f)) = f|_{E^0_{\mathrm{sg}}}$ for $f \in C_0(E^0)$ and $\pi'_0(t^n(\xi) t^n(\eta)^*) = 0$ when $n \geq 1$ (see the remark after [K1, Lemma 5.1]). The composition of Ψ and π'_0 satisfies the desired properties. □

Lemma 3.7. *We have $\pi_0^1(I_\rho \cap \mathcal{F}^1) = C_0(E^0_{\mathrm{sg}} \setminus Z)$.*

Proof. Since $I_\rho \cap \mathcal{F}^1 \supset J_\rho$, we have $\pi_0^1(I_\rho \cap \mathcal{F}^1) \supset C_0(E^0_{\mathrm{sg}} \setminus Z)$ by Lemma 3.2. We will prove the other inclusion $\pi_0^1(I_\rho \cap \mathcal{F}^1) \subset C_0(E^0_{\mathrm{sg}} \setminus Z)$. Since the restriction of the map π_0 in Lemma 3.6 to $\mathcal{F}^1$ is π_0^1, we have $\pi_0^1(I_\rho \cap \mathcal{F}^1) \subset \pi_0(I_\rho)$. For $x = t^n(\xi)at^m(\eta)^* \in I_\rho$, we have $\pi_0(x) = 0$ when $n \geq 1$ or $m \geq 1$ and

$$\pi_0(x) = (\xi|_{E^0_{\mathrm{sg}}})\pi_0^1(a)(\eta|_{E^0_{\mathrm{sg}}}) \in C_0(E^0_{\mathrm{sg}} \setminus Z)$$

when $n = m = 0$. Hence $\pi_0(I_\rho) \subset C_0(E^0_{\mathrm{sg}} \setminus Z)$. Thus we get $\pi_0^1(I_\rho \cap \mathcal{F}^1) = C_0(E^0_{\mathrm{sg}} \setminus Z)$. $\square$

Lemma 3.8. *Let $v \in X^0$. Either there exist $n \in \mathbb{N}$ and $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in Z$, or for all $n \in \mathbb{N}$ we can find $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in X^0$.*

Proof. If $v \in Z$, the first alternative holds with $n = 0$ and $e = v \in E^0$. If $v \in X^0 \setminus Z$, then $v \in X^0_{\mathrm{rg}}$ because $X^0_{\mathrm{sg}} \subset Z$. Hence we can find $e_1 \in X^1$ such that $r(e_1) = v$ and $d(e_1) \in X^0$ by Lemma 1.4. If $d(e_1) \in Z$, the first alternative holds with $n = 1$ and $e = e_1 \in E^1$. If $d(e_1) \in X^0 \setminus Z$, the argument above shows that there exists $e_2 \in X^1$ such that $r(e_2) = d(e_1)$ and $d(e_2) \in X^0$. If $d(e_2) \in Z$, the first alternative holds with $n = 2$ and $e = (e_1, e_2) \in E^2$. Repeating this argument, either we get the first alternative, or we can find $e_k \in X^1$ for $k = 1, 2, \ldots$ such that $r(e_1) = v$, $r(e_{k+1}) = d(e_k)$ and $d(e_k) \in X^0$ for all k. The latter situation implies the second alternative of this lemma. $\square$

Lemma 3.9. *We have $I_\rho \cap \mathcal{G}^0 = t^0(C_0(E^0 \setminus X^0))$.*

Proof. We have $I_\rho \cap \mathcal{G}^0 \supset J_\rho \cap \mathcal{G}^0 = t^0(C_0(E^0 \setminus X^0))$ by Lemma 3.2. We will prove the other inclusion. To the contrary, suppose that there exists $f \in C_0(E^0)$ such that $t^0(f) \in I_\rho$ and $f(v) = 1$ for some $v \in X^0$. By Lemma 3.8, either there exist $n \in \mathbb{N}$ and $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in Z$, or for all $n \in \mathbb{N}$ we can find $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in X^0$.

We first consider the case that there exist $n \in \mathbb{N}$ and $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in Z$. Take a neighborhood U of $e \in E^n$ such that the restriction of d^n to U is injective, and take $\zeta \in C_c(U)$ with $\zeta(e) = 1$. We set $g = \langle \zeta, \pi_{r^n}(f)\zeta \rangle \in C_0(E^0)$. Then we have $t^0(g) = t^n(\zeta)^* t^0(f) t^n(\zeta) \in I_\rho$ and

$$g(d^n(e)) = \sum_{e' \in (d^n)^{-1}(d^n(e))} \overline{\zeta(e')} f(r^n(e'))\zeta(e') = \overline{\zeta(e)} f(r^n(e))\zeta(e) = 1.$$

This contradicts Lemma 3.7.

Next we consider the case that for all $n \in \mathbb{N}$ we can find $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in X^0$. Take $\xi_k \in C_d(E^{n_k})$, $\eta_k \in C_d(E^{m_k})$ and $a_k \in J_\rho$ such that

$$\left\| t^0(f) - \sum_{k=1}^{K} t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^* \right\| < \frac{1}{2}.$$

Since the conditional expectation $\Psi\colon \mathcal{O}(E) \to \mathcal{F}$ defined in the proof of Lemma 3.6 satisfies $\Psi(t^0(f)) = t^0(f)$ and

$$\Psi(t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^*) = \delta_{n_k, m_k} t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^*,$$

we have

$$\left\| t^0(f) - \sum_{n_k = m_k} t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^* \right\| < \frac{1}{2}.$$

Take $n \in \mathbb{N}$ such that $n > n_k$ for k with $n_k = m_k$. We can find $e \in E^n$ with $r^n(e) = v$ and $d^n(e) \in X^0$. Take a neighborhood U of $e \in E^n$ such that the restriction of d^n to U is injective and take $\zeta \in C_c(U)$ with $0 \leq \zeta \leq 1$ and $\zeta(e) = 1$. We have $\|\zeta\| = 1$. Hence we get

$$\left\| t^n(\zeta)^* \left(t^0(f) - \sum_{n_k = m_k} t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^* \right) t^n(\zeta) \right\| < \frac{1}{2}.$$

We set $g = \langle \zeta, \pi_{r^n}(f)\zeta \rangle \in C_0(E^0)$. By Lemma 3.4, we have

$$t^n(\zeta)^* t^{n_k}(\xi_k) a_k t^{m_k}(\eta_k)^* t^n(\zeta) \in t^0(C_0(E^0 \setminus X^0))$$

for k with $n_k = m_k$. Hence $|g(d^n(e))| < 1/2$. However, we can prove $g(d^n(e)) = 1$ similarly as above. This is a contradiction.

Thus, we have shown that $I_\rho \cap \mathcal{G}^0 = t^0(C_0(E^0 \setminus X^0))$. □

Proposition 3.10. *For an admissible pair ρ, we have $\rho_{I_\rho} = \rho$.*

Proof. This follows from the equation $C_0(E^0_{\mathrm{sg}} \setminus Z_I) = \pi^1_0(I \cap \mathcal{F}^1)$ explained in the beginning of this section, Lemma 3.7 and Lemma 3.9. □

We have shown that $\rho_{I_\rho} = \rho$ for every admissible pair ρ. Next we study for which ideal I of $\mathcal{O}(E)$ the equation $I_{\rho_I} = I$ holds. Clearly the condition that I is gauge-invariant is necessary. This condition is shown to be sufficient (Proposition 3.16). For general ideals, we have the following.

Lemma 3.11. *For an ideal I of $\mathcal{O}(E)$, we have $I \cap \mathcal{F}^1 = J_{\rho_I}$ and $I \supset I_{\rho_I}$.*

Proof. Take $a \in I \cap \mathcal{F}^1$. Since the closed set Z_I is determined by $C_0(E^0_{\mathrm{sg}} \setminus Z_I) = \pi^1_0(I \cap \mathcal{F}^1)$, we have $\pi^1_0(a) \in C_0(E^0_{\mathrm{sg}} \setminus Z_I)$. For $\xi \in C_d(E^1)$, we have $\pi^1_1(a)\xi \in C_d(E^1 \setminus X^1_I)$ by Lemma 2.6 (i) because $t^1(\pi^1_1(a)\xi) = at^1(\xi) \in I$. This shows that $\pi^1_1(a) \in \ker \omega_X$. Hence $a \in J_{\rho_I}$.

Conversely take $b \in J_{\rho_I}$. Since $\pi^1_0(b) \in C_0(E^0_{\mathrm{sg}} \setminus Z_I)$, we can find $a \in I \cap \mathcal{F}^1$ such that $\pi^1_0(b) = \pi^1_0(a)$. Since $b - a \in \ker \pi^1_0$, we can find $x \in \mathcal{K}(C_d(E^1))$ with $\varphi(x) = b - a$. We have $x = \pi^1_1(b - a)$. By the former part of this proof, we have $\pi^1_1(b - a) \in \ker \omega_X$. Hence $x\xi \in C_d(E^1 \setminus X^1_I)$ for all $\xi \in C_d(E^1)$. By Lemma 2.6 (ii), we have $\varphi(x) \in I$. Hence $b = a + \varphi(x) \in I$. We have shown $I \cap \mathcal{F}^1 = J_{\rho_I}$.

This implies $I \supset J_{\rho_I}$. Since I_{ρ_I} is an ideal generated by J_{ρ_I}, we have $I \supset I_{\rho_I}$. □

Corollary 3.12. *We have $I_\rho \cap \mathcal{F}^1 = J_\rho$ for an admissible pair ρ.*

Proof. Clear from Proposition 3.10 and Lemma 3.11. □

Definition 3.13. For an admissible pair $\rho = (X^0, Z)$, we define a topological graph $E_\rho = (E^0_\rho, E^1_\rho, d_\rho, r_\rho)$ as follows. Set $Y_\rho = Z \cap X^0_{\mathrm{rg}}$ and define

$$E^0_\rho = X^0 \coprod_{\partial Y_\rho} \overline{Y_\rho}\ , \qquad E^1_\rho = X^1 \coprod_{d^{-1}(\partial Y_\rho)} d^{-1}(\overline{Y_\rho}),$$

where $\partial(Y_\rho) = \overline{Y_\rho} \setminus Y_\rho$. The domain map $d_\rho \colon E^1_\rho \to E^0_\rho$ is defined from $d \colon X^1 \to X^0$ and $d \colon d^{-1}(\overline{Y_\rho}) \to \overline{Y_\rho}$. The range map $r_\rho \colon E^1_\rho \to E^0_\rho$ is defined from $r \colon X^1 \to X^0$ and $r \colon d^{-1}(\overline{Y_\rho}) \to X^0$ (see [K2, Section 3] for the detail).

By definition, $E_\rho = X$ for $\rho = (X^0, X^0_{\mathrm{sg}})$.

Remark 3.14. For an admissible pair ρ, the four inclusions $X^0 \to E^0$, $\overline{Y_\rho} \to E^0$, $X^1 \to E^1$ and $d^{-1}(\overline{Y_\rho}) \to E^1$ define continuous proper maps $m^0\colon E^0_\rho \to E^0$ and $m^1\colon E^1_\rho \to E^1$. We can see that the pair $m = (m^0, m^1)$ is a regular factor map in the sense of [K2, Definitions 2.1 and 2.6], and that the ideal I_ρ is the kernel of the surjection $\mu\colon \mathcal{O}(E) \to \mathcal{O}(E_\rho)$ induced by $m = (m^0, m^1)$ [K2, Proposition 2.9] (see Proposition 3.16).

Proposition 3.15. *For an ideal I of $\mathcal{O}(E)$, we have a natural surjection $\mathcal{O}(E_{\rho_I}) \to \mathcal{O}(E)/I$ which is injective on the image of the map $t^0_{\rho_I}\colon C_0(E^0_{\rho_I}) \to \mathcal{O}(E_{\rho_I})$.*

Proof. It suffices to show that there exists an injective Cuntz-Krieger E_{ρ_I}-pair $\widetilde{T}$ on $\mathcal{O}(E)/I$ with $C^*(\widetilde{T}) = \mathcal{O}(E)/I$. We use the construction in [K2, Section 3] (see [K2, Remark 3.28]).

Let ω be the natural surjection from $\mathcal{O}(E)$ onto $\mathcal{O}(E)/I$ and define $T^i = \omega \circ t^i$ for $i = 0, 1$. The pair $T = (T^0, T^1)$ is a Cuntz-Krieger E-pair on $\mathcal{O}(E)/I$ satisfying that $C^*(T) = \mathcal{O}(E)/I$ and $\ker T^0 = C_0(E^0 \setminus X^0_I)$. Hence we can define an injective Toeplitz X-pair $\dot{T} = (\dot{T}^0, \dot{T}^1)$ by $\dot{T}^0(f|_{X^0}) = T^0(f)$ and $\dot{T}^1(\xi|_{X^1}) = T^1(\xi)$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$. Let $\dot{\Phi}\colon \mathcal{K}(C_d(X^1_I)) \to \mathcal{O}(E)/I$ be the $*$-homomorphism defined by $\dot{\Phi}(\theta_{\xi,\eta}) = \dot{T}^1(\xi)\dot{T}^1(\eta)^*$ for $\xi, \eta \in C_d(X^1_I)$. Then we have $\dot{\Phi}(\omega_X(x)) = \omega(\varphi(x))$ for $x \in \mathcal{K}(C_d(E^1))$. Take $g \in C_0(X^0_I)$. We will show that $\dot{T}^0(g) \in \dot{\Phi}(\mathcal{K}(C_d(X^1_I)))$ if and only if $g \in C_0(X^0_I \setminus Z_I)$. Take $f \in C_0(E^0)$ with $f|_{X^0_I} = g$. Then $\dot{T}^0(g) \in \dot{\Phi}(\mathcal{K}(C_d(X^1_I)))$ if and only if there exist $x \in \mathcal{K}(C_d(E^1))$ such that $t^0(f) - \varphi(x) \in I$. This is equivalent to $f \in C_0(E^0 \setminus Z_I)$. Hence we have shown that for $g \in C_0(X^0_I)$, $\dot{T}^0(g) \in \dot{\Phi}(\mathcal{K}(C_d(X^1_I)))$ if and only if $g \in C_0(X^0_I \setminus Z_I)$. This implies that the set Y_T defined in [K2, Definition 3.1] coincides with $Z_I \cap (X^0_I)_{\mathrm{rg}} = Y_{\rho_I}$ by [K2, Lemma 3.2]. Hence by [K2, Proposition 3.8], the Cuntz-Krieger E_{ρ_I}-pair $\widetilde{T}$ constructed as in [K2, Proposition 3.15] is injective. This completes the proof. $\square$

Proposition 3.16. *For an ideal I of $\mathcal{O}(E)$, the following are equivalent:*

(i) *I is gauge-invariant.*
(ii) *The surjection $\mathcal{O}(E_{\rho_I}) \to \mathcal{O}(E)/I$ in Proposition 3.15 is an isomorphism.*
(iii) *$I = I_{\rho_I}$.*

Proof. (i)$\Rightarrow$(ii): When I is a gauge-invariant ideal, the injective Cuntz-Krieger E_{ρ_I}-pair $\widetilde{T}$ defined in the proof of Proposition 3.15 admits a gauge action by [K2, Lemma 3.17]. Hence the Gauge Invariant Uniqueness Theorem ([K1, Theorem 4.5]) implies that the natural surjection $\mathcal{O}(E_{\rho_I}) \to \mathcal{O}(E)/I$ is an isomorphism (see also [K2, Proposition 3.27]).

(ii)$\Rightarrow$(iii): Set $J = I_{\rho_I}$ which is gauge-invariant and satisfies $\rho_J = \rho_I$ by Proposition 3.10. Hence there exists a surjection $\mathcal{O}(E_{\rho_I}) \to \mathcal{O}(E)/J$ by Proposition 3.15. Since $J \subset I$ by Lemma 3.11, there exists a surjection $\mathcal{O}(E)/J \to \mathcal{O}(E)/I$. The composition of the two surjections is nothing but the surjection in Proposition 3.15, which is an isomorphism by (ii). Hence the surjection $\mathcal{O}(E)/J \to \mathcal{O}(E)/I$ is also an isomorphism. This shows $I = J = I_{\rho_I}$.

(iii)$\Rightarrow$(i): Clear by Proposition 3.5. $\square$

Proposition 3.17. *Let I be an ideal of $\mathcal{O}(E)$. If the topological graph E_{ρ_I} is topologically free, then $\mathcal{O}(E)/I \cong \mathcal{O}(E_{\rho_I})$. Hence we have $I = I_{\rho_I}$ and I is gauge-invariant.*

Proof. When the topological graph E_{ρ_I} is topologically free, the Cuntz-Krieger Uniqueness Theorem (Proposition 6.7) implies that the surjection $\mathcal{O}(E_{\rho_I}) \to \mathcal{O}(E)/I$ in Proposition 3.15 is an isomorphism. Hence the conclusion follows from Proposition 3.16. $\square$

The following strengthens Lemma 3.11.

Proposition 3.18. *For an ideal I of $\mathcal{O}(E)$, we have $I_{\rho_I} = \bigcap_{z\in\mathbb{T}} \beta_z(I)$.*

Proof. The ideal $\bigcap_{z\in\mathbb{T}} \beta_z(I)$ is gauge-invariant, and

$$\bigcap_{z\in\mathbb{T}} \beta_z(I) \cap \mathcal{F}^1 = \bigcap_{z\in\mathbb{T}} \beta_z(I \cap \mathcal{F}^1) = I \cap \mathcal{F}^1 = J_{\rho_I}.$$

Hence $\bigcap_{z\in\mathbb{T}} \beta_z(I) = I_{\rho_I}$ by Proposition 3.16. □

We get the main result of this section.

Theorem 3.19. *The maps $I \mapsto \rho_I$ and $\rho \mapsto I_\rho$ give an inclusion reversing one-to-one correspondence between the set of all gauge-invariant ideals and the set of all admissible pairs.*

Proof. By Proposition 3.10 and Proposition 3.16, these two maps are bijections and inverses of each others.

For two gauge-invariant ideals I_1 and I_2, $I_1 \subset I_2$ if and only if $I_1 = I_1 \cap I_2$. This is equivalent to $\rho_{I_1} = \rho_{I_1\cap I_2}$. Since $\rho_{I_1\cap I_2} = \rho_{I_1} \cup \rho_{I_2}$ by Proposition 2.13, the condition $\rho_{I_1} = \rho_{I_1\cap I_2}$ is the same as $\rho_{I_1} \supset \rho_{I_2}$. Thus $I_1 \subset I_2$ if and only if $\rho_{I_1} \supset \rho_{I_2}$ for two gauge-invariant ideals I_1, I_2. □

Remark 3.20. Admissible pairs introduced in [MT, Definition 8.16] are complements of our admissible pairs, and [MT, Theorem 8.22] includes the theorem above. These theorems are generalized in [K3, Theorem 8.6] for C^*-algebras arising from general C^*-correspondences.

A discrete graph E is said to be *row-finite* if $E^0 = E^0_{\mathrm{fin}}$. For a row-finite discrete graph, the set of all gauge-invariant ideals is parameterized by invariant sets (see, for example, [BPRS, Theorem 4.1]). For a general topological graph E, the condition $E^0 = E^0_{\mathrm{fin}}$ does not suffice.

Example 3.21. Define a topological graph $E = (E^0, E^1, d, r)$ by $E^0 = E^1 = \mathbb{R}$, $d = \mathrm{id}$ and $r(x) = |x|$. Then we have $E^0 = E^0_{\mathrm{fin}}$. We have $E^0_{\mathrm{sce}} = (-\infty, 0)$ and $E^0_{\mathrm{sg}} = \overline{E^0_{\mathrm{sce}}} = (-\infty, 0]$. Take $X^0 = \{0\}$ which is a closed invariant set. We have $X^1 = \{0\}$ and a topological graph $X = (X^0, X^1, d|_{X^1}, r|_{X^1})$ has one vertex and one edge which is a loop on the vertex. We have $X^0_{\mathrm{rg}} = \{0\}$ and $X^0_{\mathrm{sg}} = \emptyset$. However $0 \in E^0_{\mathrm{sg}}$. Hence $X^0_{\mathrm{sg}} \neq E^0_{\mathrm{sg}} \cap X^0$. Thus there exists a topological graph E with $E^0 = E^0_{\mathrm{fin}}$ such that invariant subsets are not sufficient to distinguish gauge-invariant ideals of $\mathcal{O}(E)$.

The correct definition of row-finiteness for topological graphs may be the following.

Definition 3.22. A topological graph $E = (E^0, E^1, d, r)$ is said to be *row-finite* if $r(E^1) = E^0_{\mathrm{rg}}$.

It is clear that this definition of row-finiteness generalizes the one for discrete graphs. It is also clear that topological graphs defined from dynamical systems are always row-finite. Since the topological graph in Example 3.21 is not row-finite, the condition $E^0 = E^0_{\mathrm{fin}}$ does not imply the row-finiteness in general. Conversely, row-finite topological graphs need not satisfy the condition $E^0 = E^0_{\mathrm{fin}}$.

Example 3.23. Let us define a topological graph $E = (E^0, E^1, d, r)$ by $E^0 = [0, 1]$, $E^1 = (0, 1)$ and $d, r \colon E^1 \to E^0$ are the natural embedding. We have $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} = r(E^1) = (0, 1)$ and $E^0_{\mathrm{sg}} = E^0_{\mathrm{inf}} = \{0, 1\}$. Hence E is row-finite, but $E^0 \neq E^0_{\mathrm{fin}}$.

Note that $E^0 = E^0_{\mathrm{fin}}$ if and only if r is proper, and these equivalent conditions imply that the image $r(E^1)$ of r is closed. Conversely, for a row-finite topological graph E, the condition that $r(E^1)$ is closed implies $E^0 = E^0_{\mathrm{fin}}$.

Proposition 3.24. *Let E be a row-finite topological graph. For a closed invariant set $X^0 \subset E^0$, we have $X^0_{\mathrm{sg}} = X^0 \cap E^0_{\mathrm{sg}}$ and the topological graph $X = (X^0, X^1, d_X, r_X)$ is also row-finite.*

Proof. Note that a topological graph E is row-finite if and only if $E^0_{\mathrm{sg}} = E^0 \setminus r(E^1)$. Take a row-finite topological graph E, and a closed invariant set $X^0 \subset E^0$. By Proposition 2.2, we have

$$X^0_{\mathrm{sg}} \subset X^0 \cap E^0_{\mathrm{sg}} = X^0 \setminus r(E^1) \subset X^0 \setminus r_X(X^1) \subset X^0 \setminus X^0_{\mathrm{rg}} = X^0_{\mathrm{sg}}.$$

Hence we have $X^0_{\mathrm{sg}} = X^0 \cap E^0_{\mathrm{sg}}$, and the topological graph X is row-finite because $X^0_{\mathrm{sg}} = X^0 \setminus r_X(X^1)$. □

Corollary 3.25. *For a row-finite topological graph E, the map $I \mapsto X^0_I$ is a bijection from the set of all gauge-invariant ideals to the set of all closed invariant subsets of E^0.*

Proof. This follows from Theorem 3.19 and Proposition 3.24. □

4. Orbits and maximal heads

In this section, we generalize a notion of orbit spaces from ordinary dynamical systems to topological graphs, and study them.

Definition 4.1. We define a *positive orbit space* $\mathrm{Orb}^+(v)$ of $v \in E^0$ by

$$\mathrm{Orb}^+(v) = \{r^n(e) \in E^0 \mid e \in (d^n)^{-1}(v), n \in \mathbb{N}\}.$$

It is easy to see the following.

Proposition 4.2. *A subset X^0 of E^0 is positively invariant if and only if $\mathrm{Orb}^+(v) \subset X^0$ for every $v \in X^0$.*

We examine for which $v \in E^0$, $\overline{\mathrm{Orb}^+(v)}$ becomes an invariant subset of E^0.

Lemma 4.3. *If a subset X of E^0 is positively invariant or negatively invariant, then so is the closure $\overline{X}$. Hence $\overline{X}$ is invariant for an invariant subset $X \subset E^0$.*

Proof. Let X be a positively invariant subset of E^0. Take $e \in E^1$ with $d(e) \in \overline{X}$. We can find a net $\{v_\lambda\}_{\lambda \in \Lambda} \subset X$ converging to $d(e)$. Since d is locally homeomorphic, we can find a net $\{e_\lambda\} \subset E^1$ such that $\lim e_\lambda = e$ and $d(e_\lambda) = v_\lambda$ for sufficiently large λ. Since X is positively invariant and $d(e_\lambda) = v_\lambda \in X$ eventually, we have $r(e_\lambda) \in X$ eventually. Since $r(e) = \lim r(e_\lambda)$, we get $r(e) \in \overline{X}$. Thus $\overline{X}$ is positively invariant.

Let X be a negatively invariant subset of E^0. Take $v \in E^0_{\mathrm{rg}} \cap \overline{X}$. Take a compact neighborhood V of v with $V \subset E^0_{\mathrm{rg}}$. We can find a net $\{v_\lambda\}$ in V such that $v_\lambda \in X$ and $\lim v_\lambda = v$. Since $v_\lambda \in X \cap V \subset X \cap E^0_{\mathrm{rg}}$ and X is negatively invariant, for each λ there exists $e_\lambda \in E^1$ such that $r(e_\lambda) = v_\lambda$ and $d(e_\lambda) \in X$. Since $e_\lambda \in r^{-1}(V)$ for every λ and $r^{-1}(V) \subset E^1$ is compact, we get a subnet $\{e_{\lambda_i}\}$ of $\{e_\lambda\}$ which converges to some $e \in r^{-1}(V)$. We have $r(e) = \lim r(e_{\lambda_i}) = \lim v_{\lambda_i} = v$ and $d(e) = \lim d(e_{\lambda_i}) \in \overline{X}$. Hence $\overline{X}$ is negatively invariant. The latter statement is clear by the former. □

Recall that an element $e \in E^n$ for $n \geq 1$ is called a *loop* if $d^n(e) = r^n(e)$, and the vertex $d^n(e) = r^n(e)$ is called the *base point* of the loop e.

Proposition 4.4. *For any $v \in E^0$, the closed set $\overline{\mathrm{Orb}^+(v)}$ is positively invariant. The set $\overline{\mathrm{Orb}^+(v)}$ is negatively invariant whenever at least one of the following three conditions is satisfied:*

(i) $v \in E^0_{\mathrm{sg}}$.
(ii) v *is not isolated in* $\mathrm{Orb}^+(v)$.
(iii) $v \in E^0$ *is a base point of a loop.*

Proof. Set $X^0 = \overline{\mathrm{Orb}^+(v)}$. Since $\mathrm{Orb}^+(v)$ is clearly positively invariant, X^0 is positively invariant by Lemma 4.3. The set X^0 is negatively invariant if and only if $X^0_{\mathrm{sce}} \cap E^0_{\mathrm{rg}} = \emptyset$ by Proposition 2.2. For $v' \in \mathrm{Orb}^+(v) \setminus \{v\}$, there exists $e \in E^1$ such that $r(e) = v'$ and $d(e) \in \mathrm{Orb}^+(v)$. Hence we get $\mathrm{Orb}^+(v) \setminus \{v\} \subset r(X^1)$. Therefore we have $X^0_{\mathrm{sce}} = \{v\}$ when v is isolated in $\mathrm{Orb}^+(v)$ and $v \notin r(X^1)$, and we have $X^0_{\mathrm{sce}} = \emptyset$ otherwise. Hence $X^0_{\mathrm{sce}} \cap E^0_{\mathrm{rg}} \neq \emptyset$ if and only if all of the following three conditions hold;

(i) $v \in E^0_{\mathrm{rg}}$,
(ii) v is isolated in $\mathrm{Orb}^+(v)$,
(iii) $v \notin r(X^1)$.

The proof is completed if we show that $v \notin r(X^1)$ is equivalent to saying that $v \in E^0$ is a base point of no loops, under the assumption that v is isolated in $\mathrm{Orb}^+(v)$. If $v \in E^0$ is a base point of a loop, we can find $e \in E^1$ such that $r(e) = v, d(e) \in \mathrm{Orb}^+(v)$. Hence $v \in r(X^1)$. Conversely assume that $v \in r(X^1)$. Take $e \in X^1$ with $r(e) = v$. We can find a net $\{v_\lambda\} \subset \mathrm{Orb}^+(v)$ converging to $d(e) \in X^0$. Since d is locally homeomorphic, we can find a net $\{e_\lambda\} \subset E^1$ such that $\lim e_\lambda = e$ and $d(e_\lambda) = v_\lambda$ for sufficiently large λ. We have $v = \lim r(e_\lambda)$ and $r(e_\lambda) \in \mathrm{Orb}^+(v)$. Since we assume that $\{v\}$ is isolated in $\mathrm{Orb}^+(v)$, we have $v = r(e_\lambda)$ for sufficiently large λ. This means that $v \in E^0$ is a base point of a loop. The proof is completed. □

Remark 4.5. It is not difficult to see that the positive orbit space $\mathrm{Orb}^+(v)$ of $v \in E^0$ is negatively invariant if and only if either the condition (i) or (iii) in Proposition 4.4 is satisfied (see Proposition 4.10).

Next we define negative orbit spaces of vertices. An *infinite path* is a sequence $e = (e_1, e_2, \ldots, e_n, \ldots)$ with $e_k \in E^1$ and $d(e_k) = r(e_{k+1})$ for each $k = 1, 2, \ldots$. The set of all infinite paths is denoted by E^∞. For an infinite path $e = (e_1, e_2, \ldots, e_n, \ldots) \in E^\infty$, we define its range $r(e) \in E^0$ to be $r(e_1)$.

Definition 4.6. For $n \in \mathbb{N} \cup \{\infty\}$, a path $e \in E^n$ is called a *negative orbit* of $v \in E^0$ if $r^n(e) = v$ and $d^n(e) \in E^0_{\mathrm{sg}}$ when $n < \infty$.

Lemma 1.4 ensures that each $v \in E^0$ has at least one negative orbit, but v may have many negative orbits in general. If $v \in E^0_{\mathrm{sg}}$, then v itself is a negative orbit of v. We denote a negative orbit of v by $e = (e_1, e_2, \ldots, e_n) \in E^n$ for $n \in \mathbb{N} \cup \{\infty\}$. When $n = \infty$ this expression is understood as $e = (e_1, e_2, \ldots) \in E^\infty$, and when $n = 0$ this means that $e = v \in E^0$.

Definition 4.7. For each negative orbit $e = (e_1, e_2, \ldots, e_n) \in E^n$ of $v \in E^0$, a *negative orbit space* $\mathrm{Orb}^-(v, e)$ is defined by

$$\mathrm{Orb}^-(v, e) = \{v, d(e_1), d(e_2), \ldots, d(e_n)\} \subset E^0.$$

Proposition 4.8. *A subset X of E^0 is negatively invariant if and only if for each $v \in X$, there exists a negative orbit e of v such that* $\mathrm{Orb}^-(v,e) \subset X$.

Proof. Let X be a negatively invariant subset of E^0. Take $v \in X$. When $v \in E^0_{\mathrm{sg}}$, $v \in E^0$ is a negative orbit of v satisfying $\mathrm{Orb}^-(v,v) = \{v\} \subset X$. When $v \in E^0_{\mathrm{rg}}$, we can find $e_1 \in E^1$ such that $r(e_1) = v$ and $d(e_1) \in X$ because X is negatively invariant. If $d(e_1) \in E^0_{\mathrm{sg}}$, then $e_1 \in E^1$ is a negative orbit of v satisfying $\mathrm{Orb}^-(v,e_1) = \{v, d(e_1)\} \subset X$. Otherwise we get $e_2 \in E^1$ such that $r(e_2) = d(e_1)$ and $d(e_2) \in X$. Again if $d(e_2) \in E^0_{\mathrm{sg}}$, then $e = (e_1,e_2) \in E^2$ is a negative orbit of v satisfying $\mathrm{Orb}^-(v,e) = \{v, d(e_1), d(e_2)\} \subset X$. In such a manner, we get $e_k \in E^1$ for $k = 1,2,\ldots,n$ with $r(e_1) = v$ and $d(e_k) = r(e_{k+1}) \in E^0_{\mathrm{rg}} \cap X$ for $1 \leq k \leq n-1$ unless we get $d(e_n) \in E^0_{\mathrm{sg}} \cap X$. If we have $d(e_n) \in E^0_{\mathrm{sg}} \cap X$ for some $n \in \mathbb{N}$, then $e = (e_1,\ldots,e_n) \in E^n$ is a negative orbit of v with $\mathrm{Orb}^-(v,e) \subset X$. Otherwise we get $e_k \in E^1$ with $r(e_1) = v$ and $d(e_k) = r(e_{k+1}) \in X$ for $k = 1, 2, \ldots$. In this case, the infinite path $e = (e_1, e_2, \ldots) \in E^\infty$ is a negative orbit of v with $\mathrm{Orb}^-(v,e) \subset X$.

Conversely assume that for each $v \in X$, there exists a negative orbit e of v such that $\mathrm{Orb}^-(v,e) \subset X$. Take $v \in E^0_{\mathrm{rg}} \cap X$. The vertex $v \in X$ has a negative orbit $e \in E^n$ with $\mathrm{Orb}^-(v,e) \subset X$. Since $v \in E^0_{\mathrm{rg}}$, we have $n \geq 1$. Hence there exists $e_1 \in E^1$ which satisfies that $r(e_1) = v$ and $d(e_1) \in \mathrm{Orb}^-(v,e) \subset X$. Therefore X is negatively invariant. □

Definition 4.9. We define the *orbit space* $\mathrm{Orb}(v,e)$ of $v \in E^0$ with respect to a negative orbit e of v by

$$\mathrm{Orb}(v,e) = \bigcup_{v' \in \mathrm{Orb}^-(v,e)} \mathrm{Orb}^+(v').$$

For a negative orbit $e \in E^n$ of $v \in E^0$ with $0 \leq n < \infty$, we have $\mathrm{Orb}(v,e) = \mathrm{Orb}^+(d(e))$. When a negative orbit $e \in E^\infty$ of v is defined by $e = (e', e', \ldots)$ for a loop $e' \in E^*$ whose base point is v, the orbit space $\mathrm{Orb}(v,e)$ coincides with $\mathrm{Orb}^+(v)$.

Proposition 4.10. *An orbit space* $\mathrm{Orb}(v,e)$ *is an invariant set for every $v \in E^0$ and every negative orbit e of v.*

Proof. Take $v \in E^0$ and a negative orbit $e = (e_1, e_2, \ldots, e_n)$ of v ($n \in \mathbb{N} \cup \{\infty\}$). Since $\mathrm{Orb}(v,e)$ is a union of positive orbit spaces, it is positively invariant. When $n = \infty$, for every $v' \in \mathrm{Orb}(v,e)$ there exists $e' \in r^{-1}(v') \subset E^1$ with $d(e') \in \mathrm{Orb}(v,e)$. Hence $\mathrm{Orb}(v,e)$ is negatively invariant. When $n < \infty$, for every $v' \in \mathrm{Orb}(v,e)$ except $d(e)$ there exists $e' \in r^{-1}(v') \subset E^1$ with $d(e') \in \mathrm{Orb}(v,e)$. Noting that $d(e) \in E^0_{\mathrm{sg}}$, we see that $\mathrm{Orb}(v,e)$ is negatively invariant. Hence $\mathrm{Orb}(v,e)$ is invariant. □

Proposition 4.11. *A subset X of E^0 is invariant if and only if for each $v \in X$ there exists a negative orbit e of v such that* $\mathrm{Orb}(v,e) \subset X$.

Proof. Combine Proposition 4.2 and Proposition 4.8. □

Definition 4.12. A subset X^0 of E^0 is called a *maximal head* if X^0 is a non-empty closed invariant set satisfying that for any $v_1, v_2 \in X^0$ and any neighborhoods V_1, V_2 of v_1, v_2 respectively, there exists $v \in X^0$ with $\mathrm{Orb}^+(v) \cap V_1 \neq \emptyset$ and $\mathrm{Orb}^+(v) \cap V_2 \neq \emptyset$.

Equivalently, a non-empty closed invariant set X^0 is a maximal head if and only if for all $v, v' \in X^0$ there exist nets $\{e_\lambda\}, \{e'_\lambda\} \subset X^*$ of finite paths such that $d(e_\lambda) = d(e'_\lambda) \in X^0$ for all λ, $\lim r(e_\lambda) = v$ and $\lim r(e'_\lambda) = v'$.

Proposition 4.13. *For every $v \in E^0$ and every negative orbit e of v, the closed set $\overline{\mathrm{Orb}(v,e)}$ is a maximal head.*

Proof. Take $v \in E^0$ and a negative orbit $e = (e_1, \ldots, e_n) \in E^n$ of v where $n \in \mathbb{N} \cup \{\infty\}$. By Lemma 4.3 and Proposition 4.10, the set $\overline{\mathrm{Orb}(v,e)}$ is invariant. Set $v_0 = v$ and $v_k = d(e_k)$ for $k \in \{1, 2, \ldots, n\}$. Then we have $\mathrm{Orb}^-(v,e) = \{v_k\}_{k=0}^n$. Take $w_1, w_2 \in \overline{\mathrm{Orb}(v,e)}$ and neighborhoods V_1, V_2 of w_1, w_2 respectively. There exist $w_1' \in V_1 \cap \mathrm{Orb}(v,e)$ and $w_2' \in V_2 \cap \mathrm{Orb}(v,e)$. Since $\mathrm{Orb}(v,e) = \bigcup_{k=0}^n \mathrm{Orb}^+(v_k)$ and $\mathrm{Orb}^+(v_k) \subset \mathrm{Orb}^+(v_{k+1})$ for $0 \leq k \leq n-1$, we can find an integer m with $0 \leq m \leq n$ such that $w_1', w_2' \in \mathrm{Orb}^+(v_m)$. Thus $v_m \in \overline{\mathrm{Orb}(v,e)}$ satisfies that $\mathrm{Orb}^+(v_m) \cap V_1 \neq \emptyset$ and $\mathrm{Orb}^+(v_m) \cap V_2 \neq \emptyset$. Therefore $\overline{\mathrm{Orb}(v,e)}$ is a maximal head. □

The converse of Proposition 4.13 is true when E^0 is second countable.

Proposition 4.14. *If E^0 is second countable, every maximal head is of the form $\overline{\mathrm{Orb}(v,e)}$ for some negative orbit e of $v \in E^0$.*

Proof. Take a maximal head X^0 of E. Let $\{V_k\}_{k=0}^\infty$ be a countable open basis of X^0. Take a non-empty open subset V_0' of X^0 arbitrarily. Since X^0 is a maximal head, we can find $e_0 \in X^{n_0}$ and $e_0' \in X^{n_0'}$ for $n_0, n_0' \in \mathbb{N}$ such that $d(e_0) = d(e_0') \in X^0$, $r(e_0) \in V_0$ and $r(e_0') \in V_0'$. Choose compact neighborhoods $U_0 \subset X^{n_0}$ and $U_0' \subset X^{n_0'}$ of $e_0 \in X^{n_0}$ and $e_0' \in X^{n_0'}$ such that $U_0 \subset r^{-1}(V_0)$ and $U_0' \subset r^{-1}(V_0')$. Since $d(U_0) \cap d(U_0')$ is a neighborhood of $d(e_0) = d(e_0')$, we can find a non-empty open subset V_1' of X^0 such that $V_1' \subset d(U_0) \cap d(U_0')$. Inductively we can find non-empty compact sets $U_k \subset X^{n_k}$ and $U_k' \subset X^{n_k'}$ for some $n_k, n_k' \in \mathbb{N}$ such that $r(U_k) \subset V_k$ and $r(U_k') \subset d(U_{k-1}) \cap d(U_{k-1}')$ for $k \in \mathbb{N}$. We denote by U' the compact set $U_0' \times U_1' \times \cdots \times U_k' \times \cdots$. For each $k \in \mathbb{N}$, we define a closed subset F_k of the compact set U' by

$$F_k = \big\{ (e_0, \ldots, e_k, \ldots) \in U' \bigm| d(e_{j-1}) = r(e_j) \text{ for } j = 1, 2, \ldots, k \big\}$$

It is easy to see that $\{F_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of non-empty closed subsets. By the compactness of U', we can find an element $(e_0, \ldots, e_k, \ldots) \in \bigcap_{k \in \mathbb{N}} F_k$. Then $e = (e_0, \ldots, e_k, \ldots)$ is a negative orbit of $v = r(e) \in X^0$ (note that the length of e is finite when $n_k = 0$ eventually). We will prove that $X^0 = \overline{\mathrm{Orb}(v,e)}$. Since $\mathrm{Orb}^-(v,e) \subset X^0$, we have $\overline{\mathrm{Orb}(v,e)} \subset X^0$. For each $k \in \mathbb{N}$, we can find $e_k' \in E^*$ such that $d(e_k') = d(e_k) \in \mathrm{Orb}^-(v,e)$ and $r(e_k') \in V_k$. Hence we have $\mathrm{Orb}(v,e) \cap V_k \neq \emptyset$ for every $k \in \mathbb{N}$. Therefore we have $X^0 = \overline{\mathrm{Orb}(v,e)}$. □

By Proposition 4.14, for discrete graphs $E = (E^0, E^1, d, r)$ with countable E^0, every maximal head is of the form $\mathrm{Orb}(v,e)$ for some $v \in E^0$ and some negative orbit e of v. This is no longer true for discrete graphs $E = (E^0, E^1, d, r)$ such that E^0 is uncountable, as the following example shows.

Example 4.15. Let X be an uncountable set, and E^0 be the set of finite subsets of X with discrete topology. Let

$$E^1 = \big\{ (x; v) \bigm| v \in E^0 \text{ and } x \in v \big\}.$$

We define $d, r \colon E^1 \to E^0$ by $d((x;v)) = v$ and $r((x;v)) = v \setminus \{x\}$ for $(x;v) \in E^1$. For $v \in E^0$, we have $\mathrm{Orb}^+(v) = \{w \in E^0 \mid w \subset v\}$. Hence for $v_1, v_2 \in E^0$, $v_0 = v_1 \cup v_2 \in E^0$ satisfies $v_1, v_2 \in \mathrm{Orb}^+(v_0)$. This shows that E^0 is a maximal head. For a negative orbit

$e = (e_1, e_2, \dots, e_n) \in E^n$ of $v_0 \in E^0$ where $n \in \mathbb{N} \cup \{\infty\}$, $\mathrm{Orb}(v_0, e) = \{v \in E^0 \mid v \subset \Omega\}$ where $\Omega = \bigcup_{k=1}^{n} d(e_k)$. Since Ω is countable, $\mathrm{Orb}(v_0, e) \neq E^0$

Remark 4.16. In Example 13.2, we see another example of topological graphs E, which comes from a dynamical system, such that E^0 is a maximal head, but is not in the form $\overline{\mathrm{Orb}(v, e)}$ for a negative orbit e of $v \in E^0$.

5. Hereditary and saturated sets

In this section, we study the complement of invariant subsets, and get a characterization of maximal heads.

Definition 5.1. A subset V of E^0 is said to be *hereditary* if V satisfies $d(r^{-1}(V)) \subset V$, and said to be *saturated* if $v \in E^0_{\mathrm{rg}}$ satisfying $d(r^{-1}(v)) \subset V$ is in V.

It is easy to see the following.

Proposition 5.2. *For a subset X of E^0, X is positively invariant if and only if the complement $V = E^0 \setminus X$ of X is hereditary, and X is negatively invariant if and only if its complement V is saturated.*

Definition 5.3. Let us take a subset V of E^0. We define a subset $H(V)$ of E^0 by

$$H(V) = \bigcup_{n=0}^{\infty} d^n\big((r^n)^{-1}(V)\big),$$

and $S(V) \subset E^0$ by $S(V) = \bigcup_{k=0}^{\infty} V_k$, where $V_k \subset E^0$ is defined by $V_0 = V$ and

$$V_{k+1} = V_k \cup \{v \in E^0_{\mathrm{rg}} \mid d(r^{-1}(v)) \subset V_k\}.$$

Proposition 5.4. *For a subset V of E^0, $H(V)$ is the smallest hereditary set containing V. If V is open, then so is $H(V)$.*

Proof. Clear by the definition of $H(V)$. □

Proposition 5.5. *Let V be an open subset of E^0. Then $S(V)$ is open and the smallest saturated set containing V. If V is hereditary, then so is $S(V)$.*

Proof. Let $\{V_k\}$ be subsets of E^0 as in Definition 5.3 with $S(V) = \bigcup_{k=0}^{\infty} V_k$. If V is open, we can prove that V_k is open for $k \in \mathbb{N}$ inductively by Lemma 1.4. Hence $S(V)$ is open. By the definition of $S(V)$, it is clear that any saturated set containing V contains $S(V)$. We will show that $S(V)$ is saturated. Take $v \in E^0_{\mathrm{rg}}$ with $d(r^{-1}(v)) \subset S(V)$. Since $d(r^{-1}(v))$ is compact by Lemma 1.4, there exists $k \in \mathbb{N}$ such that $d(r^{-1}(v)) \subset V_k$. Hence we have $v \in V_{k+1} \subset S(V)$. Thus $S(V)$ is a saturated set containing V.

Now suppose that V is hereditary. Since we have $d(r^{-1}(V)) \subset V$, we can prove that $d(r^{-1}(V_{k+1})) \subset V_k$ for $k \in \mathbb{N}$ inductively. Hence we have $d(r^{-1}(S(V))) \subset S(V)$. We are done. □

By Proposition 5.4 and Proposition 5.5, we get the following.

Proposition 5.6. *For an open set V, the closed set $X = E^0 \setminus S(H(V))$ is the largest invariant set which does not intersect V.*

We finish this section by giving a characterization of maximal heads.

Lemma 5.7. *Let V, X be two subsets of E^0 such that X is negatively invariant. Then $S(V) \cap X \neq \emptyset$ if and only if $V \cap X \neq \emptyset$.*

Proof. Since $V \subset S(V)$, we have $S(V) \cap X \neq \emptyset$ if $V \cap X \neq \emptyset$. Conversely suppose $V \cap X = \emptyset$. Then V is contained in the complement $E^0 \setminus X$ of X which is saturated by Proposition 5.2. Hence $S(V) \subset E^0 \setminus X$. This shows $S(V) \cap X = \emptyset$. We are done. □

Lemma 5.8. *Let V_1, V_2 be two subsets of E^0, and X^0 be an invariant subset. Then $X^0 \cap S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$ if and only if $X^0 \cap H(V_1) \cap H(V_2) \neq \emptyset$.*

Proof. By Proposition 5.5, $S(H(V_2))$ is hereditary. Hence the intersection $X^0 \cap S(H(V_2))$ is negatively invariant. Thus Lemma 5.7 shows $X^0 \cap S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$ if and only if $X^0 \cap H(V_1) \cap S(H(V_2)) \neq \emptyset$. By the same reason, $X^0 \cap H(V_1) \cap S(H(V_2)) \neq \emptyset$ if and only if $X^0 \cap H(V_1) \cap H(V_2) \neq \emptyset$. We are done. □

Proposition 5.9. *An invariant set X^0 of E^0 is a maximal head if and only if $X_1^0 \cup X_2^0 \supset X^0$ implies either $X_1^0 \supset X^0$ or $X_2^0 \supset X^0$ for two invariant sets X_1^0, X_2^0.*

Proof. Let X^0 be a maximal head. Take invariant sets X_1^0 and X_2^0 satisfying $X_1^0 \not\supset X^0$ and $X_2^0 \not\supset X^0$, and we will show that $X_1^0 \cup X_2^0 \not\supset X^0$. There exist $v_1, v_2 \in X^0$ with $v_1 \notin X_1^0$ and $v_2 \notin X_2^0$. Since X^0 is a maximal head, there exist $v \in X^0$ and $e_1, e_2 \in E^*$ with $d(e_1) = d(e_2) = v$, $r(e_1) \notin X_1^0$ and $r(e_2) \notin X_2^0$. Since X_1^0 and X_2^0 are positively invariant, we have $v \notin X_1^0$ and $v \notin X_2^0$. Therefore, $X_1^0 \cup X_2^0 \not\supset X^0$.

Conversely assume that a closed invariant set X^0 satisfies that $X_1^0 \cup X_2^0 \supset X^0$ implies either $X_1^0 \supset X^0$ or $X_2^0 \supset X^0$ for two invariant sets X_1^0, X_2^0. Take $v_1, v_2 \in X^0$ and neighborhoods V_1, V_2 of v_1, v_2 respectively. For $i = 1, 2$, set $X_i^0 = E^0 \setminus S(H(V_i))$ which are invariant by Proposition 5.6. We have $X_1^0 \not\supset X^0$ and $X_2^0 \not\supset X^0$. By the assumption, we have $X_1^0 \cup X_2^0 \not\supset X^0$. This implies $X^0 \cap S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$. By Lemma 5.8, we have $X^0 \cap H(V_1) \cap H(V_2) \neq \emptyset$. Then $v \in X^0 \cap H(V_1) \cap H(V_2)$ satisfies $\mathrm{Orb}^+(v) \cap V_1 \neq \emptyset$ and $\mathrm{Orb}^+(v) \cap V_2 \neq \emptyset$. This shows that X^0 is a maximal head. □

6. Ideals and topological freeness

Proposition 6.1. *For an open subset V of E^0, the ideal I of $\mathcal{O}(E)$ generated by $t^0(C_0(V)) \subset \mathcal{O}(E)$ is gauge-invariant and satisfies $\rho_I = \big(E^0 \setminus S(H(V)), E^0_{\mathrm{sg}} \setminus S(H(V))\big)$.*

Proof. Since $t^0(C_0(V))$ is invariant under the gauge action, the ideal I is gauge-invariant. Set $X^0 = E^0 \setminus S(H(V))$ which is a closed invariant set. We set $\rho = (X^0, E^0_{\mathrm{sg}} \cap X^0)$ which is an admissible pair. We will show $\rho_I = \rho$. Since $V \subset S(H(V)) = E^0 \setminus X^0$, we have

$$t^0(C_0(V)) \subset t^0(C_0(E^0 \setminus X^0)) \subset I_\rho.$$

Thus $I \subset I_\rho$. By Lemma 2.9 and Proposition 3.10, we have $\rho_I \supset \rho_{I_\rho} = \rho$. Thus $X_I^0 \supset X^0$ and $Z_I \supset E^0_{\mathrm{sg}} \cap X^0$. Since $t^0(C_0(V)) \subset I$, we have $X_I^0 \subset E^0 \setminus V$. Thus Proposition 5.6 shows $X_I^0 \subset E^0 \setminus S(H(V)) = X^0$. Therefore $X_I^0 = X^0$. Hence we get $Z_I = E^0_{\mathrm{sg}} \cap X^0$. Thus we have $\rho_I = \rho$. We are done. □

Remark 6.2. For an open subset V of E^0, the condition $d(r^{-1}(V)) \subset V$ considered in [K2, Proposition 5.9] is nothing but the hereditariness, and the condition that each $v \in E^0 \setminus V$ is regular and satisfies $d^n((r^n)^{-1}(v)) \subset V$ for some $n \in \mathbb{N}$ is equivalent to $S(V) = E^0$. Thus Proposition 6.1 shows that this condition is a necessary and sufficient condition for A_V to be full, as predicted in [K2, Remark 5.10].

Now we recall some arguments from [K2, Section 5]. Let F^0 be a hereditary open subset of E^0, and define $F^1 = r^{-1}(F^0)$. The set F^1 is an open subset of E^1, and satisfies $d(F_1), r(F_1) \subset F^0$. Thus $F = (F^0, F^1, d|_{F^1}, r|_{F^1})$ is a subgraph of E in the sense of [K2, Definition 5.1].

Proposition 6.3. *Under the notations above, the C^*-subalgebra A of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_d(F^1))$ is a hereditary and full subalgebra of the ideal I generated by $t^0(C_0(F^0))$, and there exists a natural isomorphism between A and $\mathcal{O}(F)$ which commutes with the natural injections from $C_0(F^0)$ and $C_d(F^1)$ to A and $\mathcal{O}(F)$.*

Proof. For $\xi \in C_c(F^1)$, we can find $f \in C_0(F^0)$ such that $\pi_r(f)\xi = \xi$. Hence $t^1(\xi) = t^0(f)t^1(\xi) \in I$. This shows $t^1(C_d(F^1)) \subset I$. Therefore A is a full subalgebra of I. The rest of the statements follows from [K2, Propositions 5.5, 5.9]. □

By the two propositions above, we get the following proposition.

Proposition 6.4. *Let I be a gauge-invariant ideal of $\mathcal{O}(E)$ with $Z_I = E^0_{\mathrm{sg}} \cap X^0_I$. Take a hereditary open subset F^0 with $S(F^0) = E^0 \setminus X^0_I$ (for example take $F^0 = E^0 \setminus X^0_I$). Set a subgraph $F = (F^0, F^1, d|_{F^1}, r|_{F^1})$ of E by $F^1 = r^{-1}(F^0)$. Then I is strongly Morita equivalent to $\mathcal{O}(F)$.*

Proof. By Proposition 6.1 and Theorem 3.19, I coincides with the ideal generated by $t^0(C_0(F^0))$, which is strongly Morita equivalent to $\mathcal{O}(F)$ by Proposition 6.3. □

In [K5], we will see that for an arbitrary gauge-invariant ideal I of $\mathcal{O}(E)$, one can find a topological graph F such that I is strongly Morita equivalent to $\mathcal{O}(F)$. We will also see that a gauge-invariant ideal I itself can be expressed as $\mathcal{O}(F')$ for some topological graph F' which is less natural than the topological graph F above. It is hopeless (or useless) to express an ideal of $\mathcal{O}(E)$ which is not gauge-invariant as a C^*-algebra of some topological graph (see Example 6.9).

In the rest of this section, we apply Proposition 6.3 to prove that topological freeness is needed in the Cuntz-Krieger Uniqueness Theorem ([K1, Theorem 5.12]). We recall the definition of topological freeness, and the statement of the theorem.

Definition 6.5. A loop $e = (e_1, \ldots, e_n)$ is said to be *simple* if $r(e_i) \neq r(e_j)$ for $i \neq j$, and said to be *without entrances* if $r^{-1}(r(e_k)) = \{e_k\}$ for $k = 1, \ldots, n$.

It is easy to see that if $v \in E^0$ is a base point of a loop, then v is a base point of a simple loop. It is also easy to see that if $v \in E^0$ is a base point of a loop without entrances, then there exists a unique simple loop whose base point is v, and this loop is also without entrances. (A loop e without entrances is in the form $e = (e', \ldots, e')$ where e' is a simple loop without entrances.)

Definition 6.6 ([K1, Definition 5.4])**.** A topological graph E is said to be *topologically free* if the set of base points of loops without entrances has an empty interior.

The following theorem is called the Cuntz-Krieger Uniqueness Theorem.

Proposition 6.7 ([K1, Theorem 5.12])**.** *Let E be a topologically free topological graph. Then the natural surjection $\mathcal{O}(E) \to C^*(T)$ is an isomorphism for every injective Cuntz-Krieger E-pair T.*

We will prove its converse. Namely, when E is not topologically free, we will find an injective Cuntz-Krieger E-pair T such that the natural surjection $\mathcal{O}(E) \to C^*(T)$ is not an isomorphism. To find such injective Cuntz-Krieger E-pair T, it suffices to find a non-zero ideal I of $\mathcal{O}(E)$ with $\bigcap_{z\in\mathbb{T}} \beta_z(I) = 0$ by the following lemma.

Lemma 6.8. *For an ideal I of the C^*-algebra $\mathcal{O}(E)$, the following conditions are equivalent:*

(i) *I is the kernel of the $*$-homomorphism $\mathcal{O}(E) \to C^*(T)$ induced by an injective Cuntz-Krieger E-pair T.*
(ii) *$I \cap t^0(C_0(E^0)) = 0$.*
(iii) *$X_I^0 = E^0$.*
(iv) *$\rho_I = (E^0, E^0_{\mathrm{sg}})$.*
(v) *$\bigcap_{z\in\mathbb{T}} \beta_z(I) = 0$.*

Proof. Clearly we have (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iv). The equivalence (iv) $\iff$ (v) follows from Proposition 3.18. □

We start with the following example.

Example 6.9. Let n be a positive integer. Let $E_n = (E_n^0, E_n^1, d_n, r_n)$ be a discrete graph such that $E_n^0 = E_n^1 = \mathbb{Z}/n\mathbb{Z}$, $d_n = \mathrm{id}_{\mathbb{Z}/n\mathbb{Z}}$, and $r_n(k) = k+1$. The graph E_n consists of one loop without entrances. Thus every vertex of E_n is a base point of a loop without entrances. Therefore E_n is not topologically free.

Let us denote by $\{1, 2, \ldots, n\}$ the elements of $\mathbb{Z}/n\mathbb{Z}$. The matrix units of the C^*-algebra $\mathbb{M}_n$ of all $n \times n$ matrices are denoted by $\{u_{k,l}\}_{k,l\in\mathbb{Z}/n\mathbb{Z}}$, and the generating unitary of $C(\mathbb{T})$ is denote by $w \in C(\mathbb{T})$. We will prove that $\mathcal{O}(E_n) \cong C(\mathbb{T}) \otimes \mathbb{M}_n$ (cf. the proof of Proposition 8.7). We define a $*$-homomorphism $T^0 \colon C(E_n^0) \to C(\mathbb{T}) \otimes \mathbb{M}_n$ and a linear map $T^1 \colon C_{d_n}(E_n^1) \to C(\mathbb{T}) \otimes \mathbb{M}_n$ by

$$T^0(f) = \sum_{k\in\mathbb{Z}/n\mathbb{Z}} f(k) \otimes u_{k,k}, \quad T^1(\xi) = \sum_{k=1}^{n-1} \xi(k) \otimes u_{k+1,k} + (\xi(n)w) \otimes u_{1,n},$$

for $f \in C(E_n^0)$ and $\xi \in C_{d_n}(E_n^1)$. It is routine to check that the pair $T = (T^0, T^1)$ is an injective Cuntz-Krieger E_n-pair which admits a gauge action and satisfies $C^*(T) = C(\mathbb{T}) \otimes \mathbb{M}_n$. Hence by [K1, Theorem 4.5], we have $\mathcal{O}(E_n) \cong C(\mathbb{T}) \otimes \mathbb{M}_n$.

Lemma 6.10. *Let n be a positive integer, and E_n be the discrete graph in Example 6.9. Then there exists a non-zero ideal I of $\mathcal{O}(E_n)$ with $I \cap t^0(C(E_n^0)) = 0$.*

Proof. Take $z \in \mathbb{T}$. Then the ideal I of $\mathcal{O}(E)$ corresponding to the ideal $C_0(\mathbb{T} \setminus \{z\}) \otimes \mathbb{M}_n$ of $C(\mathbb{T}) \otimes \mathbb{M}_n \cong \mathcal{O}(E_n)$ satisfies that $I \neq 0$ and $I \cap t^0(C(E_n^0)) = 0$. □

Lemma 6.11. *Let $E_n = (E_n^0, E_n^1, d_n, r_n)$ be the discrete graph in Example 6.9, and X be a locally compact space. Let $F = (F^0, F^1, d_F, r_F)$ be a topological graph such that there exist homeomorphisms $m^0 \colon F^0 \to E_n^0 \times X$ and $m^1 \colon F^1 \to E_n^1 \times X$ with $(d_n \times \mathrm{id}_X) \circ m^1 = m^0 \circ d_F$ and $(r_n \times \mathrm{id}_X) \circ m^1 = m^0 \circ r_F$. Then there exists a non-zero ideal I' of $\mathcal{O}(F)$ with $I' \cap t_F^0(C_0(F^0)) = 0$ where $t_F = (t_F^0, t_F^1)$ is the universal Cuntz-Krieger F-pair on $\mathcal{O}(F)$.*

Proof. It is easy to see that the two maps m^0, m^1 induce isomorphisms $C_0(F^0) \cong C(E_n^0) \otimes C_0(X)$ and $C_{d_F}(F^1) \cong C_{d_n}(E_n^1) \otimes C_0(X)$, and these isomorphisms induce an isomorphism

$\mathcal{O}(F) \cong \mathcal{O}(E_n) \otimes C_0(X)$ (see [K2, Proposition 7.7]). By Lemma 6.10, there exists a non-zero ideal I of $\mathcal{O}(E_n)$ such that $I \cap t^0(C(E_n^0)) = 0$. Then the ideal I' of $\mathcal{O}(F)$ corresponding to the ideal $I \otimes C_0(X)$ of $\mathcal{O}(E_n) \otimes C_0(X)$ satisfies the desired conditions. □

Proposition 6.12. *A topological graph E is not topologically free if and only if there exist a non-empty open subset V of E^0 and a positive integer n such that all vertices in V are base points of simple loops in E^n without entrances, and that $\sigma = r \circ (d|_{r^{-1}(V)})^{-1}$ is a well-defined continuous map from V to V with $\sigma^n = \mathrm{id}_V$.*

Proof. If there exist a non-empty open subset V of E^0 and a positive integer n such that all vertices in V are base points of simple loops in E^n without entrances, then E is not topologically free by definition.

Suppose that E is not topologically free. Then we can find a non-empty open set W_0 such that all vertices in W_0 are base points of loops without entrances. Set $V_0 = H(W_0)$. It is not difficult to see that V_0 is a non-empty open set with $d(r^{-1}(V_0)) = V_0$ such that all vertices in V_0 are base points of loops without entrances. We will show that the restriction of d to $r^{-1}(V_0)$ is an injection onto V_0. Take $e_1, e_1' \in r^{-1}(V_0)$ with $d(e_1) = d(e_1')$. Let $(e_1, \ldots, e_n)$ be a simple loop without entrances whose base point is $r(e_1) \in V_0$, and $(e_1', \ldots, e_m')$ the one of $r(e_1') \in V_0$. We may assume that $n \geq m$ without loss of generality. Since $r(e_2) = d(e_1) = d(e_1') = r(e_2')$, we have $e_2 = e_2'$. We also have $e_3 = e_3'$ because $r(e_3) = d(e_2) = d(e_2') = r(e_3')$. Recursively, we can see that $e_k = e_k'$ for $k = 2, 3, \ldots, m$. If $n > m$, then we have $e_{m+1} = e_1'$, which is impossible because $d(e_1) = d(e_1')$ and $(e_1, \ldots, e_n)$ is a simple loop. Hence we have $n = m$ and $e_1 = e_1'$. Thus the restriction of d to $r^{-1}(V_0)$ is a bijection onto V_0. Since d is a local homeomorphism, its restriction to the open set $r^{-1}(V_0)$ is a homeomorphism from $r^{-1}(V_0)$ to V_0. Hence we can define a continuous map $\sigma\colon V_0 \to V_0$ by $\sigma = r \circ (d|_{r^{-1}(V_0)})^{-1}$. By Baire's category theorem (see, for example, [T1, Proposition 2.2]), there exist a non-empty open subset V of V_0 and a positive integer n such that every vertex $v \in V$ satisfies $\sigma^k(v) \neq v$ for $k = 1, 2, \ldots, n-1$ and $\sigma^n(v) = v$. This shows that $v \in V$ is a base point of a simple loop in E^n without entrances. The proof is completed. □

Proposition 6.13. *If E is not topologically free, then there exists a non-zero ideal I of $\mathcal{O}(E)$ with $\bigcap_{z \in \mathbb{T}} \beta_z(I) = 0$.*

Proof. By Proposition 6.12, there exist a non-empty open subset V of E^0 and a positive integer n such that all vertices in V are base points of simple loops in E^n without entrances, and $\sigma = r \circ (d|_{r^{-1}(V)})^{-1}$ is a well-defined continuous map from V to V with $\sigma^n = \mathrm{id}_V$. Take $v \in V$. We can find a compact neighborhood V' of v such that $\sigma^k(v) \notin V'$ for $k = 1, 2, \ldots, n-1$. Take an open set V'' such that $v \in V'' \subset V'$ and define $X = V'' \setminus \bigcup_{k=1}^{n-1} \sigma^k(V')$ which is a non-empty open subset of E^0. We have $X \cap \sigma^k(X) = \emptyset$ for $k = 1, 2, \ldots, n-1$. Define $F^0 = \bigcup_{k=0}^{n-1} \sigma^k(X)$. Then F^0 is a hereditary open subset of E^0. Define a subgraph $F = (F^0, F^1, d|_{F^1}, r|_{F^1})$ of E such that $F^1 = r^{-1}(F^0)$. We can apply Lemma 6.11 to the topological graph F, and hence get a non-zero ideal I' of $\mathcal{O}(F)$ with $I' \cap t_F^0(C_0(F^0)) = 0$ where $t_F^0\colon C_0(F^0) \to \mathcal{O}(F)$ is the natural injection. By Proposition 6.3, there exists a natural isomorphism from $\mathcal{O}(F)$ to the C^*-subalgebra A of $\mathcal{O}(E)$ generated by $t^0(C_0(F^0))$ and $t^1(C_d(F^1))$ which commutes with the natural injections from $C_0(F^0)$ and $C_d(F^1)$ to $\mathcal{O}(F)$ and A. Let $I'' \subset A$ be the image of I' under this natural isomorphism. Thus I'' is an ideal of A. By Lemma 6.8, we have $\bigcap_{z \in \mathbb{T}} \beta_z'(I') = 0$ where β' is the gauge action of $\mathcal{O}(F)$. Since the natural isomorphism $\mathcal{O}(F) \to A \subset \mathcal{O}(E)$ preserves

the images of $C_0(F^0)$ and $C_d(F^1)$, it is equivariant under the two gauge actions. Hence we get $\bigcap_{z\in\mathbb{T}}\beta_z(I'')=0$. By Proposition 6.3, A is a hereditary and full subalgebra of the ideal J generated by $t^0(C_0(F^0))$. Since the map $I\mapsto I\cap A$ is a bijection from the set of ideals of J to the ones of A, we can find an ideal I of J with $I\cap A=I''$. The ideal $\bigcap_{z\in\mathbb{T}}\beta_z(I)$ of J satisfies

$$\bigcap_{z\in\mathbb{T}}\beta_z(I)\cap A=\bigcap_{z\in\mathbb{T}}\beta_z(I\cap A)=\bigcap_{z\in\mathbb{T}}\beta_z(I'')=0.$$

Hence we get $\bigcap_{z\in\mathbb{T}}\beta_z(I)=0$. Thus we get a non-zero ideal I of $\mathcal{O}(E)$ that satisfies $\bigcap_{z\in\mathbb{T}}\beta_z(I)=0$. □

Now the following theorem strengthens the Cuntz-Krieger Uniqueness Theorem.

Theorem 6.14. *The following conditions for a topological graph E are equivalent:*

(i) *E is topologically free.*
(ii) *The natural surjection $\mathcal{O}(E)\to C^*(T)$ is an isomorphism for every injective Cuntz-Krieger E-pair T.*
(iii) *$\rho_I=(E^0,E^0_{\mathrm{sg}})$ implies $I=0$.*
(iv) *Any non-zero ideal I of $\mathcal{O}(E)$ satisfies $I\cap t^0(C_0(E^0))\neq 0$.*

Proof. Clear from Proposition 6.7, Lemma 6.8 and Proposition 6.13. □

Proposition 6.15. *If a topological graph E is not topologically free, then there exist non-zero ideals I_1,I_2 of $\mathcal{O}(E)$ such that $I_1\cap I_2=0$.*

Proof. We can easily see the conclusion when E is the topological graph E_n in Example 6.9. For general topological graphs which are not topologically free, we can prove it similarly to the proofs of Lemma 6.11 and Proposition 6.13. □

7. Free topological graphs

In this section, we give the condition on topological graphs E so that all ideals of $\mathcal{O}(E)$ are gauge-invariant.

Definition 7.1. For a positive integer n, we denote by $\mathrm{Per}_n(E)$ the set of vertices v satisfying the following three conditions;

(i) there exists a simple loop $(l_1,\ldots,l_n)\in E^n$ whose base point is v,
(ii) if $e\in E^1$ satisfies $d(e)\in\mathrm{Orb}^+(v)$ and $r(e)=r(l_k)$ for some k, then we have $e=l_k$,
(iii) v is isolated in $\mathrm{Orb}^+(v)$.

We set $\mathrm{Per}(E)=\bigcup_{n=1}^\infty\mathrm{Per}_n(E)$ and $\mathrm{Aper}(E)=E^0\setminus\mathrm{Per}(E)$.

An element in $\mathrm{Per}(E)$ is called a *periodic point* while an element in $\mathrm{Aper}(E)$ is called an *aperiodic point*. The conditions (i) and (ii) above mean that $v\in E^0$ is a base point of exactly one simple loop, and the condition (iii) says that there exists no "approximated loop" whose base point is v. When topological graphs come from homeomorphisms, these notions coincide with the ordinary ones in dynamical systems (see, for example, [T1, T2]). Note that $\bigcup_{k=1}^n\mathrm{Per}_k(E)$ is not necessarily closed unlike the case of ordinary dynamical systems. If E is not topologically free, then there exists a non-empty open subset U of E^0 such that every vertex in U is a base point of a loop without entrances. Thus we have $U\subset\mathrm{Per}(E)$. Hence if $\overline{\mathrm{Aper}(E)}=E^0$, then E is topologically free. The converse is not true in general (consider discrete graphs).

Definition 7.2. A topological graph E is said to be *free* if $\operatorname{Aper}(E) = E^0$.

This definition is a generalization of freeness in ordinary dynamical systems. This is also a generalization of *Condition K* in the theory of graph algebras (see, for example, [KPRR]). In [MT, Definition 9.1], Muhly and Tomforde define Condition K for a topological quiver, which coincides with our freeness by Proposition 7.5 (see also [MT, Proposition 9.9]). Free topological graphs are topologically free. In fact, we get a stronger statement (Proposition 7.5).

Lemma 7.3. *Let us take $v \in \operatorname{Per}_n(E)$. Let $(l_1, \ldots, l_n) \in E^n$ be the unique simple loop whose base point is v. Then we have the following.*

- (i) *The closed set $X^0 = \overline{\operatorname{Orb}^+(v)}$ is invariant.*
- (ii) *$r(l_k)$ is isolated in X^0 for $k = 1, \ldots, n$.*
- (iii) *In the topological graph $X = (X^0, X^1, d|_{X^1}, r|_{X^1})$, the loop $(l_1, \ldots, l_n) \in X^n$ has no entrances.*
- (iv) *The topological graph X is not topologically free.*

Proof. (i) Since v is the base point of the loop $(l_1, \ldots, l_n) \in E^n$, Proposition 4.4 implies that X^0 is invariant.

(ii) By the assumption, $r(l_1) = v$ is isolated in X^0. Take a net $\{v_\lambda\}$ in $\operatorname{Orb}^+(v)$ which converges to $r(l_2)$, and we will show that $v_\lambda = r(l_2)$ eventually. Since $r(l_2) = d(l_1)$ and d is locally homeomorphic, we can find a net $\{e_\lambda\} \subset E^1$ such that $\lim e_\lambda = l_1$ and $d(e_\lambda) = v_\lambda$. We have $\lim r(e_\lambda) = r(l_1)$ and $r(e_\lambda) \in \operatorname{Orb}^+(v)$. Since $r(l_1)$ is isolated in X^0, we have $r(e_\lambda) = r(l_1)$ eventually. For such λ, we have $e_\lambda = l_1$ by the condition (ii) in Definition 7.1. Therefore we have $v_\lambda = r(l_2)$ eventually. This proves that $r(l_2)$ is isolated in X^0. Recursively we can prove that $r(l_k)$ is isolated in X^0 for $k = 3, \ldots, n$.

(iii) Take $e \in X^1$ with $r(e) = r(l_k)$ for some $k \in \{1, 2, \ldots, n\}$, and we will prove $e = l_k$. Since $d(e) \in X^0$, there exists a net $\{v_\lambda\} \subset \operatorname{Orb}^+(v)$ converging to $d(e)$. Since d is locally homeomorphic, we can find a net $\{e_\lambda\} \subset E^1$ such that $\lim e_\lambda = e$ and $d(e_\lambda) = v_\lambda$ eventually. We have $\lim r(e_\lambda) = r(e) = r(l_k)$ and $r(e_\lambda) \in \operatorname{Orb}^+(v)$. Since $r(l_k)$ is isolated in X^0 by (ii), we have $r(e_\lambda) = r(l_k)$ eventually. By the condition (ii) in Definition 7.1, we have $e_\lambda = l_k$ eventually. Hence we have $e = l_k$.

(iv) Since $\{v\}$ is an open subset of X^0, the proof completes by (iii).

□

For $v \in \operatorname{Per}_n(E)$, Lemma 7.3 (ii) implies that $r(l_k) \in \operatorname{Per}_n(E)$, for the unique simple loop $(l_1, \ldots, l_n) \in E^n$ whose base point is v.

Recall that for an admissible pair $\rho = (X^0, Z)$ we define a topological graph E_ρ in Definition 3.13, and we have $E_\rho = X$ for $\rho = (X^0, X^0_{\mathrm{sg}})$.

Lemma 7.4. *If there exists an admissible pair $\rho = (X^0, Z)$ such that E_ρ is not topologically free, then E is not free.*

Proof. Let us take an open subset V of $E^0_\rho = X^0 \amalg_{\partial Y_\rho} \overline{Y_\rho}$ such that every $v \in V$ is a base point of a loop without entrances. Since every vertex in $E^0_\rho \setminus X^0$ is a source, V is contained in X^0. Take $v \in V$ arbitrarily and let $(l_1, \ldots, l_n) \in E^n_\rho$ be the unique simple loop without entrances whose base point is v. We will prove that $v \in V \subset E^0$ is in $\operatorname{Per}_n(E)$. Since $d(e) \in X^0 \subset E^0_\rho$ implies $e \in X^1 \subset E^1_\rho$ for $e \in E^1_\rho$, we have $l_k \in X^1$ for $k = 1, \ldots, n$.

Let us take $e \in E^1$ with $d(e) \in \mathrm{Orb}^+(v)$ and $r(e) = r(l_k)$ for some $k \in \{1, \ldots, n\}$. Since $d(e) \in \mathrm{Orb}^+(v) \subset X^0$, we have $e \in X^1 \subset E^1_\rho$. Since the loop $(l_1, \ldots, l_n)$ has no entrances in the topological graph E_ρ, we have $e = l_k$. Thus the condition (ii) in Definition 7.1 is satisfied. We will show that v is isolated in $\mathrm{Orb}^+(v)$. Let us take a net $\{v_\lambda\}$ in $\mathrm{Orb}^+(v)$ which converges to v. Since $v \in V$, $\mathrm{Orb}^+(v) \subset X^0$ and V is open in X^0, $v_\lambda \in V$ eventually. For such λ, v_λ is a base point of a loop in X^n without entrances. Therefore since there exists a path from v to v_λ, we can find a path from v_λ to v. Hence $v_\lambda = r(l_k)$ for some $k \in \{1, \ldots, n\}$. Thus we get $v_\lambda = v$ eventually. This implies that v is isolated in $\mathrm{Orb}^+(v)$. Hence we have $v \in \mathrm{Per}_n(E)$ and so $\mathrm{Per}(E) \neq \emptyset$. Therefore E is not free. $\square$

Proposition 7.5. *A topological graph E is free if and only if E_ρ is topologically free for every admissible pair ρ.*

Proof. This follows from Lemma 7.3 and Lemma 7.4. $\square$

Theorem 7.6. *For a topological graph E, every ideal of $\mathcal{O}(E)$ is gauge-invariant if and only if E is free. Thus if E is free, the set of all ideals corresponds bijectively to the set of all admissible pairs by the maps $I \mapsto \rho_I$ and $\rho \mapsto I_\rho$.*

Proof. By Proposition 3.16 and Theorem 6.14, every ideal of $\mathcal{O}(E)$ is gauge-invariant if and only if E_ρ is topologically free for every admissible pair. It is equivalent to the freeness of E by Proposition 7.5. The latter statement follows from the former and Theorem 3.19. $\square$

8. Minimal topological graph and simplicity of $\mathcal{O}(E)$

We give a couple of conditions on E all of which are equivalent to saying that $\mathcal{O}(E)$ becomes simple. We start with a detailed analysis on topological graphs with a periodic point.

Let E be a topological graph, and v_0 be a periodic point of E. Let $l = (l_1, l_2, \ldots, l_n)$ be the unique simple loop whose base point is v_0. We set $X^0 = \overline{\mathrm{Orb}^+(v_0)}$ which is a closed invariant set of E^0. We define a topological graph $X = (X^0, X^1, d_X, r_X)$ so that $X^1 = d^{-1}(X^0)$ and d_X, r_X are the restrictions of d, r to X^1. We simply write d, r for d_X, r_X. This causes no confusion because in the sequel we only deal with the topological graph X, and do not use the topological graph E. We set $V = \{d(l_1), d(l_2), \ldots, d(l_n)\}$ which is an open subset of X^0.

Lemma 8.1. *For an infinite path $e = (e_1, e_2, \ldots) \in X^\infty$ of the graph X, the following conditions are equivalent:*

(i) *There exists k such that $d(e_k) \in V$.*
(ii) *There exists k_0 such that $d(e_k) \in V$ for every $k \geq k_0$.*
(iii) *e is in the form $(e', l, l, \ldots)$ with some $e' \in X^*$.*

Proof. This follows from the fact that $l = (l_1, l_2, \ldots, l_n)$ is a loop without entrances in X. $\square$

We denote by Λ_l the set of infinite paths $e \in X^\infty$ satisfying the equivalent conditions in Lemma 8.1.

The set V coincides with $H(\{v_0\})$ considered in the topological graph X. Let $F^0 = S(V) \subset X^0$. Since V is open in X^0, F^0 is an open hereditary and saturated subset of

X^0 by Proposition 5.5. We define a subgraph F of X by $F = (F^0, F^1, d|_{F^1}, r|_{F^1})$ where $F^1 = r^{-1}(F^0)$.

Proposition 8.2. *For $v \in X^0$, the following are equivalent:*

(i) $v \in F^0$.
(ii) $\{v\}$ *is open and every negative orbit of* v *is in* Λ_l.
(iii) $\{v\}$ *is open and has only finitely many negative orbits.*
(iv) $\{v\}$ *is open and the set* $\{e \in \Lambda_l \mid r(e) = v\}$ *is finite.*

Proof. (i)$\Rightarrow$(ii): Let W be the set of $v \in X^0$ satisfying (ii). Clearly $V \subset W$. We will show that W is saturated. Take $v \in X^0_{\mathrm{rg}}$ with $d(r^{-1}(v)) \subset W$. It is clear that every negative orbit of v is in Λ_l. Since W is open, Lemma 1.4 gives a neighborhood V' of v such that $r^{-1}(V') \subset d^{-1}(W)$. By replacing V' with a smaller set, we may assume that V' is a compact neighborhood of v with $V' \subset X^0_{\mathrm{rg}}$. Then $r^{-1}(V')$ is compact and satisfies $r(r^{-1}(V')) = V'$ (see [K1, Proposition 2.8]). Since $d^{-1}(W)$ is a discrete set, $r^{-1}(V')$ is a finite set. Hence V' is also finite, and this shows that $\{v\}$ is open. Thus $v \in W$. We have shown that W is a saturated set containing V. By Proposition 5.5, we have $F^0 \subset W$.

(ii)$\Rightarrow$(iii): Take $v \in X^0$ whose negative orbits are in Λ_l. We have $v \in X^0_{\mathrm{rg}}$. Hence $U_1 = r^{-1}(v)$ is a compact set. We also have $d(U_1) \subset X^0_{\mathrm{rg}}$. Since $r\colon r^{-1}(X^0_{\mathrm{rg}}) \to X^0_{\mathrm{rg}}$ is a proper map, $U_2 = r^{-1}(d(U_1))$ is a compact set. Hence the subsets $U_k \subset X^1$, defined by $U_{k+1} = r^{-1}(d(U_k))$ recursively, are compact. The set of negative orbits of v coincides with the compact set

$$\Omega = \big\{e = (e_1, e_2, \ldots) \;\big|\; e_k \in U_k, d(e_k) = r(e_{k+1})\big\} \subset U_1 \times U_2 \times \cdots .$$

For $k = 1, 2, \ldots,$ we define a subset Ω_k of Ω by

$$\Omega_k = \{e = (e_1, e_2, \ldots) \in \Omega \mid d(e_k) \in V\}.$$

For $e = (e_1, e_2, \ldots) \in \Omega$, $d(e_k) \in V$ implies $d(e_{k+1}) \in V$. Hence we have $\Omega_k \subset \Omega_{k+1}$. We also see that Ω_k coincides with $\{e \in E^k \mid r^k(e) = v, d^k(e) \in V\}$. Since V is discrete and $(r^k)^{-1}(v)$ is compact, this set is a finite set. Hence Ω_k is also a finite set. Since V is open, Ω_k is an open subset of Ω for every k. By (ii), we have $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. By the compactness of Ω, we have $\Omega = \Omega_k$ for some k. This shows that Ω is finite. Thus the set of negative orbits of $\{v\}$ is a finite set.

(iii)$\Rightarrow$(iv): Obvious.

(iv)$\Rightarrow$(i): Take $v \in X^0$ satisfying (iv). We first show that v' also satisfies (iv) for $v' \in d(r^{-1}(v))$. It is clear that the set $\{e' \in \Lambda_l \mid r(e') = v'\}$ is finite for all $v' \in d(r^{-1}(v))$. Since $\{v\}$ is open, $d(r^{-1}(v)) \subset X^0$ is open. By the condition (iv), $d(r^{-1}(v)) \cap \mathrm{Orb}^+(v_0)$ is finite. Since $\mathrm{Orb}^+(v_0)$ is dense in X^0, $d(r^{-1}(v))$ is a finite subset of $\mathrm{Orb}^+(v_0)$. This shows that $\{v'\}$ is open for every $v' \in d(r^{-1}(v))$. Hence v' also satisfies (iv) for all $v' \in d(r^{-1}(v))$. Next we show $v \in X^0_{\mathrm{rg}}$ for $v \in X^0$ satisfying (iv). Since $\{v\}$ is open in X^0, we have $v \in \mathrm{Orb}^+(v_0)$. Hence the set $r^{-1}(v)$ is non-empty. Since we have already seen that $d(r^{-1}(v))$ is a finite subset, the non-empty set $r^{-1}(v)$ is finite. This shows $v \in X^0_{\mathrm{rg}}$ by [K1, Proposition 2.8].

For $e = (e_1, e_2, \ldots) \in \Lambda_l$, we define $|e| \in \mathbb{N}$ by $|e| = 0$ if $r(e) \in V$ and $|e| = \max\{k \mid d(e_k) \notin V\}$ otherwise. For v satisfying (iv), we define $n(v) \in \mathbb{N}$ by

$$n(v) = \max\big\{|e| \in \mathbb{N} \;\big|\; e \in \Lambda_l \text{ with } r(e) = v\big\}.$$

We will show $v \in F^0$ by induction on $n(v)$. It is clear when $n(v) = 0$. Suppose that we have shown $v \in F^0$ for v satisfying (iv) with $n(v) \leq k$. Take v satisfying (iv) with $n(v) = k+1$. As we saw above, each $v' \in d(r^{-1}(v))$ satisfies (iv). Clearly $n(v') \leq k$ for $v' \in d(r^{-1}(v))$. Hence we have $d(r^{-1}(v)) \subset F^0$ by the assumption of the induction. Since $v \in X^0_{\mathrm{rg}}$ and F^0 is saturated, we have $v \in F^0$. We are done. □

Corollary 8.3. *We have $F^0 \subset X^0_{\mathrm{rg}}$, and F^0 is a discrete subset of X^0.*

Proposition 8.2 motivates the following definition.

Definition 8.4. We say that a topological graph E is *generated by a loop* $l = (l_1, l_2, \ldots, l_n)$ if E^0 is discrete and every negative orbit is in the form $(e', l, l, \ldots) \in E^\infty$ with some $e' \in E^*$.

We have already seen the following.

Proposition 8.5. *Let E be a topological graph. For $v \in \mathrm{Per}(E)$, the topological graph F defined above is generated by the loop l.*

Every topological graph generated by a loop arises in this way.

Proposition 8.6. *Let E be a topological graph generated by a loop $l = (l_1, l_2, \ldots, l_n)$. Then the base point v_0 of the loop l is in $\mathrm{Per}_n(E)$, and $E^0 = \mathrm{Orb}^+(v_0) = S(H(\{v_0\}))$.*

Proof. It is clear from the definition that l has no entrances. Hence $v_0 \in \mathrm{Per}_n(E)$. It is also clear to see $E^0 = \mathrm{Orb}^+(v_0)$. Now $E^0 = S(H(\{v_0\}))$ follows from Proposition 8.2. □

For a Hilbert space H, we denote by $K(H)$ (resp. $B(H)$) the C^*-algebra of all compact operators (resp. all bounded operators) on H.

Proposition 8.7. *Let E be a topological graph generated by a loop. Then $\mathcal{O}(E)$ is isomorphic to $C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$.*

Proof. Let E be a topological graph generated by a loop $l = (l_1, l_2, \ldots, l_n)$. We denote by $l^\infty \in E^\infty$ the infinite path $(l, l, \ldots)$. Let us denote by $w \in C(\mathbb{T})$ the generating unitary of $C(\mathbb{T})$ defined by $w(z) = z$ for $z \in \mathbb{T}$. Let us denote by $\{u_{e,e'}\}_{e,e' \in E^\infty}$ the canonical matrix units of $K(\ell^2(E^\infty))$. By Proposition 8.2, for $v \in E^0$ the set $E^\infty_v = \{e \in E^\infty \mid r(e) = v\}$ is finite. Since E^0 and E^1 are discrete, the linear span of the characteristic functions $\{\delta_v\}_{v \in E^0}$ is dense in $C_0(E^0)$, and the one of $\{\delta_e\}_{e \in E^1}$ is dense in $C_d(E^1)$. We define a $*$-homomorphism $T^0 \colon C_0(E^0) \to C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$ and a linear map $T^1 \colon C_d(E^1) \to C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$ so that for $v \in E^0$ and $e \in E^1 \setminus \{l_1\}$,

$$T^0(\delta_v) = 1 \otimes \sum_{e' \in E^\infty_v} u_{e',e'}, \quad T^1(\delta_e) = 1 \otimes \sum_{e' \in E^\infty_{d(e)}} u_{(e,e'),e'}$$

and $T^1(\delta_{l_1}) = w \otimes u_{(l_1,l_2,\ldots),(l_2,l_3,\ldots)}$. It is routine to check

$$T^1(\delta_e)^* T^1(\delta_{e'}) = \delta_{e,e'} T^0(\delta_{d(e)}), \quad T^0(\delta_v) T^1(\delta_e) = \delta_{v,r(e)} T^1(\delta_e)$$
$$T^0(\delta_v) = \sum_{e \in r^{-1}(v)} T^1(\delta_e) T^1(\delta_e)^*$$

for $e, e' \in E^1$ and $v \in E^0$. This shows that $T = (T^0, T^1)$ is a Cuntz-Krieger E-pair. Since $E^\infty_v \neq \emptyset$ for all $v \in E^0$, T is injective. We will show that T admits a gauge action. For $e = (e_1, e_2, \ldots) \in E^\infty$, we define $|e| \in \mathbb{N}$ by $|e| = \min\{k \mid e_k = l_1\}$. For $z \in \mathbb{T}$, we define an automorphism σ_z of $C(\mathbb{T})$ by $\sigma_z(f)(z') = f(z^n z')$ for $z' \in \mathbb{T}$, and an automorphism

$\mathrm{Ad}(u_z)$ of $K(\ell^2(E^\infty))$ by $\mathrm{Ad}(u_z)(x) = u_z x u_z^*$ where $u_z \in B(\ell^2(E^\infty))$ is a unitary defined by $u_z = \sum_{e\in E^\infty} z^{|e|} u_{e,e}$ which converges strongly. Then the automorphism $\beta'_z = \sigma_z \otimes \mathrm{Ad}(u_z)$ of $C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$ satisfies

$$\beta'_z(1 \otimes u_{e,e'}) = z^{|e|-|e'|}(1 \otimes u_{e,e'}), \quad \beta'_z(w \otimes u_{l^\infty,l^\infty}) = z^n(w \otimes u_{l^\infty,l^\infty}).$$

Thus the action β' is a gauge action for the pair T. Hence T induces an injective map $\mathcal{O}(E) \to C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$. We will show that it is surjective. For $e = (e_1, e_2, \ldots) \in E^\infty$, we have

$$T^1(\delta_{e_1})T^1(\delta_{e_2}) \cdots T^1(\delta_{e_{k-1}})T^0(\delta_{d(l_1)}) = 1 \otimes u_{e,l^\infty}$$

where $k = |e|$. We also have

$$T^1(\delta_{l_1})T^1(\delta_{l_2}) \cdots T^1(\delta_{l_n}) = w \otimes u_{l^\infty,l^\infty}.$$

Since $C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$ is generated by $\{1 \otimes u_{e,l^\infty}\}_{e\in E^\infty}$ and $w \otimes u_{l^\infty,l^\infty}$, we see that the map $\mathcal{O}(E) \to C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$ is surjective. Therefore $\mathcal{O}(E)$ is isomorphic to $C(\mathbb{T}) \otimes K(\ell^2(E^\infty))$. □

From now on, we study for which topological graph E, the C^*-algebra $\mathcal{O}(E)$ is simple. We introduce the following notion.

Definition 8.8. A topological graph E is said to be *minimal* if there exist no closed invariant sets other than $\emptyset$ or E^0.

Proposition 8.9. *For a topological graph E, the following conditions are equivalent:*

(i) *E is minimal.*
(ii) *The orbit space $\mathrm{Orb}(v, e)$ is dense in E^0 for every $v \in E^0$ and every negative orbit e of v.*
(iii) *For every non-empty open set $V \subset E^0$, we have $S(H(V)) = E^0$.*

Proof. (i)⇒(ii): For a negative orbit e of $v \in E^0$, the closed subset $\overline{\mathrm{Orb}(v, e)}$ is invariant. By (i), we have $\overline{\mathrm{Orb}(v, e)} = E^0$.

(ii)⇒(i): Let X^0 be a non-empty closed invariant subset of E^0. Take $v \in X^0$. By Proposition 4.11, there exists a negative orbit e of v such that $\mathrm{Orb}(v, e) \subset X^0$. By (ii), we have $\overline{\mathrm{Orb}(v, e)} = E^0$. Hence $X^0 = E^0$. Thus E is minimal.

(i)⇒(iii): Take a non-empty open set $V \subset E^0$. Then $E^0 \setminus S(H(V))$ is a closed invariant subset such that $E^0 \setminus S(H(V)) \neq E^0$. By the minimality, we have $E^0 \setminus S(H(V)) = \emptyset$. Thus we have $S(H(V)) = E^0$.

(iii)⇒(i): Let X^0 be a closed invariant subset of E^0 with $X^0 \neq E^0$. Then $V = E^0 \setminus X^0$ is a non-empty hereditary and saturated set. By (iii), we have $V = S(H(V)) = E^0$. Hence we have $X^0 = \emptyset$. Thus E is minimal. □

Remark 8.10. The notion of minimality extends the one of ordinary dynamical systems for which Proposition 8.9 is well-known. When a graph is discrete, the condition (ii) above is equivalent to the cofinality in the sense of [BPRS] with the extra condition that for two vertices $v \in E^0$ and $w \in E^0_{\mathrm{sg}}$ there exists a path $e \in E^*$ such that $d(e) = w$ and $r(e) = v$. For a discrete graph, the equivalence of (ii) and (iii) in Proposition 8.9 had been certainly known (see, for example, Introduction of [P]).

Lemma 8.11. *A topological graph generated by a loop is minimal.*

Proof. Let E be a topological graph generated by a loop l, and $v_0 \in E^0$ be the base point of the loop l. By definition, for any $v \in E^0$ and any negative orbit e of v, the negative orbit space $\mathrm{Orb}^-(v, e)$ contains v_0. Since $\mathrm{Orb}^+(v_0) = E^0$, E is minimal by Proposition 8.9. $\square$

Theorem 8.12. *For a topological graph E, the following conditions are equivalent:*
 (i) *The C^*-algebra $\mathcal{O}(E)$ is simple.*
 (ii) *E is minimal and topologically free.*
 (iii) *E is minimal and free.*
 (iv) *E is minimal and not generated by a loop.*

Proof. (i)⇒(iv): If E is not minimal, then there exists a non-trivial gauge-invariant ideal. Hence $\mathcal{O}(E)$ is not simple. If E is generated by a loop, $\mathcal{O}(E)$ is not simple by Proposition 8.7.

(iv)⇒(iii): Suppose that E is minimal and not free. Take $v_0 \in \mathrm{Per}(E)$. Since E is minimal, $\overline{\mathrm{Orb}^+(v_0)} = E^0$ by Proposition 8.9. Hence $\{v_0\}$ is open in E^0. Using Proposition 8.9 again, we get $S(H(\{v_0\})) = E^0$. Hence E is generated by a loop by Proposition 8.5.

(iii)⇒(ii): Obvious.

(ii)⇒(i): Assume that E is minimal and topologically free. Take an ideal I of $\mathcal{O}(E)$ with $I \neq \mathcal{O}(E)$. Then X_I is a closed invariant set other than $\emptyset$. By the minimality, we have $X_I = E^0$. By Theorem 6.14 we have $I = 0$. Thus $\mathcal{O}(E)$ is simple. $\square$

Corollary 8.13. *When E^0 is not discrete, $\mathcal{O}(E)$ is simple if and only if E is minimal.*

Remark 8.14. The above theorem generalizes the result on simplicity of graph algebras ([S, Theorem 12], [DT, Corollary 2.15], see also [P, Theorem 4]). In the case that d is injective, a topological graph $E = (E^0, E^1, d, r)$ is generated by a loop if and only if E is minimal and E^0 is a finite set. Hence in this case the condition (iv) in the above theorem is equivalent to

(iv)' E is minimal and E^0 is an infinite set.

Thus Theorem 8.12 generalizes a criterion for simplicity of homeomorphism C^*-algebras due to Zeller-Meier [Z].

9. Primeness for admissible pairs

In this section, we give a necessary condition for an ideal to be prime in terms of admissible pairs. We will use it after in order to determine all prime ideals (Theorem 11.14). Recall that an ideal I of a C^*-algebra A is said to be *prime* if for two ideals I_1, I_2 of A, $I_1 \cap I_2 \subset I$ implies either $I_1 \subset I$ or $I_2 \subset I$. We define primeness for admissible pairs.

Definition 9.1. An admissible pair ρ is called *prime* if $\rho_1 \cup \rho_2 \supset \rho$ implies either $\rho_1 \supset \rho$ or $\rho_2 \supset \rho$ for two admissible pairs $\rho_1,\ \rho_2$.

It is well-known that an ideal I is prime if and only if the equality $I_1 \cap I_2 = I$ implies either $I_1 = I$ or $I_2 = I$ for two ideals I_1, I_2 (see the proof of (iii)⇒(iv) of Proposition 9.2). The following is the counterpart of this fact for prime admissible pairs.

Proposition 9.2. *For an admissible pair ρ, the following are equivalent:*
 (i) *ρ is prime.*
 (ii) *For two admissible pairs $\rho_1,\ \rho_2$, the equality $\rho_1 \cup \rho_2 = \rho$ implies either $\rho_1 = \rho$ or $\rho_2 = \rho$.*

(iii) *For two gauge invariant ideals* I_1, I_2 *of* $\mathcal{O}(E)$*, the equality* $I_1 \cap I_2 = I_\rho$ *implies either* $I_1 = I_\rho$ *or* $I_2 = I_\rho$*.*
(iv) *For two gauge invariant ideals* I_1, I_2 *of* $\mathcal{O}(E)$*, the inclusion* $I_1 \cap I_2 \subset I_\rho$ *implies either* $I_1 \subset I_\rho$ *or* $I_2 \subset I_\rho$*.*

Proof. (i)$\Rightarrow$(ii): Take two admissible pairs ρ_1, ρ_2 with $\rho_1 \cup \rho_2 = \rho$. By (i), we have either $\rho_1 \supset \rho$ or $\rho_2 \supset \rho$. Hence we get either $\rho_1 = \rho$ or $\rho_2 = \rho$.

(ii)$\Rightarrow$(iii): Take two gauge invariant ideals I_1, I_2 with $I_1 \cap I_2 = I_\rho$. We have $\rho_{I_1} \cup \rho_{I_2} = \rho$. By (ii), we have either $\rho_{I_1} = \rho$ or $\rho_{I_2} = \rho$. By Proposition 3.16, we have either $I_1 = I_\rho$ or $I_2 = I_\rho$.

(iii)$\Rightarrow$(iv): Take two gauge invariant ideals I_1, I_2 with $I_1 \cap I_2 \subset I_\rho$. Then we have

$$(I_1 + I_\rho) \cap (I_2 + I_\rho) = (I_1 \cap I_2) + I_\rho = I_\rho.$$

By (iii), either $I_1 + I_\rho = I_\rho$ or $I_2 + I_\rho = I_\rho$ holds. Hence we get either $I_1 \subset I_\rho$ or $I_2 \subset I_\rho$.

(iv)$\Rightarrow$(i): Take two admissible pairs ρ_1, ρ_2 with $\rho_1 \cup \rho_2 \supset \rho$. The two gauge-invariant ideals I_{ρ_1} and I_{ρ_2} satisfy

$$I_{\rho_1} \cap I_{\rho_2} = I_{\rho_1 \cup \rho_2} \subset I_\rho.$$

Hence we have either $I_{\rho_1} \subset I_\rho$ or $I_{\rho_2} \subset I_\rho$. By Theorem 3.19, we get either $\rho_1 \supset \rho$ or $\rho_2 \supset \rho$. Thus ρ is prime. $\square$

We will use the implication (ii)$\Rightarrow$(i) to determine which admissible pairs are prime.

Proposition 9.3. *If an ideal* I *of* $\mathcal{O}(E)$ *is prime, then* ρ_I *is a prime admissible pair.*

Proof. The proof goes similarly as the proof of (iv)$\Rightarrow$(i) in Proposition 9.2, hence we omit it. $\square$

In general, the converse of Proposition 9.3 is not true (see Propositions 11.1 and 11.3). We will determine all prime admissible pairs (Proposition 9.8).

Lemma 9.4. *If an admissible pair* $\rho = (X^0, Z)$ *is prime, then either* $Z = X^0_{\mathrm{sg}}$ *or* $Z = X^0_{\mathrm{sg}} \cup \{v\}$ *for some* $v \in X^0_{\mathrm{rg}}$*.*

Proof. Let $\rho = (X^0, Z)$ be a prime admissible pair. To derive a contradiction, assume $Z \setminus X^0_{\mathrm{sg}}$ has two elements v_1, v_2. Take open sets $V_1 \ni v_1$, $V_2 \ni v_2$ with $V_1 \cap V_2 = \emptyset$, $V_1 \cap X^0_{\mathrm{sg}} = \emptyset$ and $V_2 \cap X^0_{\mathrm{sg}} = \emptyset$. Then $\rho_i = (X^0, Z \setminus V_i)$ $(i = 1, 2)$ are admissible pairs satisfying $\rho = \rho_1 \cup \rho_2$. However, we have $\rho \not\subset \rho_1$ and $\rho \not\subset \rho_2$. This contradicts the primeness of ρ. $\square$

First we consider the case $Z = X^0_{\mathrm{sg}}$.

Lemma 9.5. *An admissible pair* $\rho = (X^0, X^0_{\mathrm{sg}})$ *is prime if and only if* X^0 *is a maximal head.*

Proof. Suppose that $\rho = (X^0, X^0_{\mathrm{sg}})$ is a prime admissible pair. Take invariant sets X^0_1, X^0_2 with $X^0 \subset X^0_1 \cup X^0_2$. Set $\rho_i = (X^0_i, X^0_i \cap E^0_{\mathrm{sg}})$ for $i = 1, 2$. Since

$$X^0_{\mathrm{sg}} \subset X^0 \cap E^0_{\mathrm{sg}} = (X^0_1 \cap E^0_{\mathrm{sg}}) \cup (X^0_2 \cap E^0_{\mathrm{sg}}),$$

We have $\rho \subset \rho_1 \cup \rho_2$. Since ρ is prime, either $\rho \subset \rho_1$ or $\rho \subset \rho_2$ holds. Hence we have either $X^0 \subset X^0_1$ or $X^0 \subset X^0_2$. By Proposition 5.9, X^0 is a maximal head. Conversely assume that X^0 is a maximal head. Take two admissible pairs $\rho_1 = (X^0_1, Z_1)$, $\rho_2 = (X^0_2, Z_2)$ with $\rho_1 \cup \rho_2 = \rho$. By Proposition 5.9, either $X^0 \subset X^0_1$ or $X^0 \subset X^0_2$ holds. We may assume

$X^0 \subset X_1^0$. Then $X^0 = X_1^0$. Hence $X_{\mathrm{sg}}^0 = (X_1^0)_{\mathrm{sg}} \subset Z_1 \subset X_{\mathrm{sg}}^0$. We get $\rho_1 = \rho$. By Proposition 9.2, $\rho = (X^0, X_{\mathrm{sg}}^0)$ is a prime admissible pair. $\square$

Next we consider the case $Z = X_{\mathrm{sg}}^0 \cup \{v\}$.

Lemma 9.6. *An admissible pair $\rho = (X^0, X_{\mathrm{sg}}^0 \cup \{v\})$ is prime for some $v \in X_{\mathrm{rg}}^0 \cap E_{\mathrm{sg}}^0$ if and only if $X^0 = \overline{\mathrm{Orb}^+(v)}$.*

Proof. Suppose that an admissible pair $\rho = (X^0, X_{\mathrm{sg}}^0 \cup \{v\})$ is prime. Since $v \in E_{\mathrm{sg}}^0$, the pair $\rho_1 = (\overline{\mathrm{Orb}^+(v)}, \overline{\mathrm{Orb}^+(v)} \cap E_{\mathrm{sg}}^0)$ is an admissible pair by Proposition 4.4. If we set an admissible pair $\rho_2 = (X^0, X_{\mathrm{sg}}^0)$, then ρ_1 and ρ_2 satisfy $\rho \subset \rho_1 \cup \rho_2$. Since ρ is prime and $\rho \not\subset \rho_2$, we have $\rho \subset \rho_1$. Hence $\overline{\mathrm{Orb}^+(v)} \subset X^0 \subset \overline{\mathrm{Orb}^+(v)}$. Thus, we get $X^0 = \overline{\mathrm{Orb}^+(v)}$. Conversely, assume $X^0 = \overline{\mathrm{Orb}^+(v)}$. Take two admissible pairs $\rho_1 = (X_1^0, Z_1)$, $\rho_2 = (X_2^0, Z_2)$ with $\rho_1 \cup \rho_2 = \rho$. We may assume $v \in Z_1$. Then we have $X^0 = \overline{\mathrm{Orb}^+(v)} \subset X_1^0 \subset X^0$. Hence $X_1^0 = X^0$. We have $X_{\mathrm{sg}}^0 \cup \{v\} = (X_1^0)_{\mathrm{sg}} \cup \{v\} \subset Z_1 \subset X_{\mathrm{sg}}^0 \cup \{v\}$. Therefore $\rho_1 = \rho$. By Proposition 9.2, ρ is a prime admissible pair. $\square$

Note that if $v \in E_{\mathrm{sg}}^0$ then $X^0 = \overline{\mathrm{Orb}^+(v)}$ is invariant by Proposition 4.4. We define

$$BV(E) = \{v \in E_{\mathrm{sg}}^0 \mid v \in X_{\mathrm{rg}}^0 \text{ where } X^0 = \overline{\mathrm{Orb}^+(v)}\}.$$

Elements in $BV(E)$ are called *breaking vertices*. When E is discrete, breaking vertices are infinite receivers ([BHRS]). In general, breaking vertices may not be in E_{inf}^0.

Example 9.7. Let E be the topological graph in Example 3.21. The vertex $0 \in \mathbb{R}$ is a breaking vertex. We have $E_{\mathrm{inf}}^0 = \emptyset$ and $0 \in \overline{E_{\mathrm{sce}}^0}$.

For $v \in BV(E)$, we define $\rho_v = (X^0, X_{\mathrm{sg}}^0 \cup \{v\})$ where $X^0 = \overline{\mathrm{Orb}^+(v)}$. We denote by $\mathcal{M}(E)$ the set of all maximal heads. For $X^0 \in \mathcal{M}(E)$, we define $\rho_{X^0} = (X^0, X_{\mathrm{sg}}^0)$.

Proposition 9.8. *The map $x \mapsto \rho_x$ gives a bijection from $BV(E) \amalg \mathcal{M}(E)$ to the set of all prime admissible pairs.*

Proof. The injectivity of the map is easy to see from the definitions, and the surjectivity follows from Lemma 9.4, Lemma 9.5 and Lemma 9.6. $\square$

10. Primeness of $\mathcal{O}(E)$

A C^*-algebra is said to be *prime* if 0 is a prime ideal. Using the results in the previous section, we give conditions on E for $\mathcal{O}(E)$ to be a prime C^*-algebra.

Definition 10.1. A topological graph E is called *topologically transitive* if we have $H(V_1) \cap H(V_2) \neq \emptyset$ for any two non-empty open sets $V_1, V_2 \subset E^0$.

There exist many equivalent conditions of topological transitivity.

Proposition 10.2. *For a topological graph E, consider the following conditions.*

(i) *E is topologically transitive.*
(ii) *For two non-empty open sets $V_1, V_2 \subset E^0$, we have $S(H(V_1)) \cap S(H(V_2)) \neq \emptyset$.*
(iii) *The admissible pair (E^0, E_{sg}^0) is prime.*
(iv) *The set E^0 is a maximal head.*
(v) *There exists $v \in E^0$ and a negative orbit e of v such that $\mathrm{Orb}(v, e)$ is dense in E^0.*

Then the conditions (i), (ii), (iii) *and* (iv) *are equivalent and implied by* (v). *When* E^0 *is second countable, the five conditions are equivalent.*

Proof. Lemma 5.8 gives (i)$\Leftrightarrow$(ii). By the definition of maximal heads, we have (i)$\Leftrightarrow$(iv). By Lemma 9.5, we have (iii)$\Leftrightarrow$(iv). By Proposition 4.13, (v) implies (iv). When E^0 is second countable, we have (iv)$\Rightarrow$(v) by Proposition 4.14. $\square$

By this proposition, we can see that a minimal topological graph is topologically transitive.

Theorem 10.3. *A C^*-algebra $\mathcal{O}(E)$ is prime if and only if E is topologically free and topologically transitive.*

Proof. Suppose that the C^*-algebra $\mathcal{O}(E)$ is prime. By Proposition 6.15, E is topologically free. The admissible pair $\rho_0 = (E^0, E^0_{\mathrm{sg}})$ is prime by Proposition 9.3. Hence E is topologically transitive. Conversely assume that E is topologically free and topologically transitive. Take two ideals I_1, I_2 of $\mathcal{O}(E)$ with $I_1 \cap I_2 = 0$, and we will show that either $I_1 = 0$ or $I_2 = 0$. Since $I_1 \cap I_2 = 0$, we have $\rho_{I_1} \cup \rho_{I_1} = \rho_0 = (E^0, E^0_{\mathrm{sg}})$. By Proposition 10.2, (E^0, E^0_{sg}) is prime. Hence either $(E^0, E^0_{\mathrm{sg}}) = \rho_{I_1}$ or $(E^0, E^0_{\mathrm{sg}}) = \rho_{I_2}$ holds. We have either $I_1 = 0$ or $I_2 = 0$ by Theorem 6.14. Thus we show that $\mathcal{O}(E)$ is prime. $\square$

In Proposition 11.3, we will see that a topologically transitive topological graph is not topologically free only when there exists $v \in \mathrm{Per}(E)$ such that $E^0 = \overline{\mathrm{Orb}^+(v)}$.

11. Prime ideals

In this section, we completely determine the set of prime ideals of the C^*-algebra $\mathcal{O}(E)$ of a topological graph E. In Proposition 9.3, we see that for a prime ideal P of $\mathcal{O}(E)$, the admissible pair ρ_P is necessarily prime. The following proposition determines when the converse of this fact is true for gauge-invariant ideals.

Proposition 11.1. *For a prime admissible pair ρ, I_ρ is a prime ideal if and only if E_ρ is topologically free.*

Proof. If E_ρ is not topologically free, then I_ρ is not prime by Proposition 6.15. Suppose E_ρ is topologically free. Take two ideals I_1, I_2 of $\mathcal{O}(E)$ with $I_1 \cap I_2 = I_\rho$. By Proposition 3.10 and Proposition 2.13, we have $\rho = \rho_{I_\rho} = \rho_{I_1 \cap I_2} = \rho_{I_1} \cup \rho_{I_2}$. Since ρ is a prime admissible pair, either $\rho_{I_1} = \rho$ or $\rho_{I_2} = \rho$ by Proposition 9.2. Without loss of generality, we may assume $\rho_{I_1} = \rho$. Then we have $E_{\rho_{I_1}} = E_\rho$ which is topologically free. Hence by Proposition 3.17 we have $I_1 = I_\rho$. Thus I_ρ is a prime ideal. $\square$

We study which prime admissible pair ρ satisfies that E_ρ is topologically free. Recall that in Proposition 9.8 we saw that all prime admissible pairs are in the form $\rho_v = (\overline{\mathrm{Orb}^+(v)}, \overline{\mathrm{Orb}^+(v)}_{\mathrm{sg}} \cup \{v\})$ for $v \in BV(E)$, or $\rho_{X^0} = (X^0, X^0_{\mathrm{sg}})$ for $X^0 \in \mathcal{M}(E)$.

Proposition 11.2. *For $v \in BV(E)$, the topological graph E_{ρ_v} is topologically free.*

Proof. From Definition 3.13, we see that $E^0_{\rho_v}$ is the disjoint union of $\overline{\mathrm{Orb}^+(v)}$ and an extra vertex $\bar{v}$. Since the vertex $\bar{v}$ receives no edge, it is not a base point of a loop. For every vertex $w \in \mathrm{Orb}^+(v) \subset E^0_{\rho_v}$, there exists $e \in E^*_{\rho_v}$ such that $d(e) = \bar{v}$ and $r(e) = w$. Hence w cannot be a base point of a loop without entrances. Since $\mathrm{Orb}^+(v) \cup \{\bar{v}\}$ is dense in $E^0_{\rho_v}$, the set of base points of loops without entrances has an empty interior. Thus the topological graph E_{ρ_v} is topologically free. $\square$

We denote

$$\mathcal{M}_{\mathrm{per}}(E) = \big\{\overline{\mathrm{Orb}^+(v)} \in \mathcal{M}(E) \;\big|\; v \in \mathrm{Per}(E)\big\},$$

and $\mathcal{M}_{\mathrm{aper}}(E) = \mathcal{M}(E) \setminus \mathcal{M}_{\mathrm{per}}(E)$.

Proposition 11.3. *For $X^0 \in \mathcal{M}(E)$, the topological graph $E_{\rho_{X^0}} = X$ is topologically free if and only if $X^0 \in \mathcal{M}_{\mathrm{aper}}(E)$.*

Proof. If $X^0 = \overline{\mathrm{Orb}^+(v)}$ for some $v \in \mathrm{Per}(E)$, then X is not topologically free by Lemma 7.3 (iv). Suppose that X^0 is a maximal head such that the topological graph X is not topologically free, and we will show that $X^0 \in \mathcal{M}_{\mathrm{per}}(E)$. By Proposition 6.12, there exist a non-empty open subset V of X^0 and a positive integer n such that all vertices in V are base points of simple loops in X^n without entrances, and that $\sigma = r \circ (d|_{r^{-1}(V)})^{-1}$ is a well-defined continuous map from V to V with $\sigma^n = \mathrm{id}_V$. Take $v \in V$ arbitrarily. We will show that $V = \{v, \sigma(v), \ldots, \sigma^{n-1}(v)\}$. Take $w \in V$. Since X^0 is a maximal head, we have two nets $\{e_\lambda\}, \{e'_\lambda\} \subset X^*$ such that $d(e_\lambda) = d(e'_\lambda) \in X^0$ and $\lim r(e_\lambda) = v$ and $\lim r(e'_\lambda) = w$. We may assume that $r(e_\lambda), r(e'_\lambda) \in V$ for every λ. Since $r(e'_\lambda) \in V$ is a base point of a simple loop in X^n without entrances, we can find a path from $r(e'_\lambda)$ to $d(e'_\lambda) = d(e_\lambda)$. Thus there exists a path from $r(e'_\lambda)$ to $r(e_\lambda)$. Since $r(e_\lambda) \in V$ is a base point of a simple loop in X^n without entrances, we can find $k_\lambda \in \{0, 1, \ldots, n-1\}$ such that $r(e'_\lambda) = \sigma^{k_\lambda}(r(e_\lambda))$ for each λ. Then, we can find $k \in \{0, 1, \ldots, n-1\}$ with $k_\lambda = k$ frequently. We have

$$w = \lim r(e'_\lambda) = \lim \sigma^k(r(e_\lambda)) = \sigma^k(\lim r(e_\lambda)) = \sigma^k(v).$$

Thus we have shown that $V = \{v, \sigma(v), \ldots, \sigma^{n-1}(v)\}$. Since V is open in X^0, $\{v\}$ is also open in X^0. Hence $\{v\}$ is isolated in $\mathrm{Orb}^+(v)$ because $\mathrm{Orb}^+(v) \subset X^0$, Since the condition (ii) in Definition 7.1 is clearly satisfied, we have $v \in \mathrm{Per}_n(E)$. The proof will complete once we will show that $X^0 = \overline{\mathrm{Orb}^+(v)}$. Clearly $X^0 \supset \overline{\mathrm{Orb}^+(v)}$. Take $w \in X^0$. By noting that $\{v\}$ is a neighborhood of $v \in X^0$, we can find a net $\{e_\lambda\} \subset X^*$ such that $\lim r(e_\lambda) = w$ and $v \in \mathrm{Orb}^+(d(e_\lambda))$ because X^0 is a maximal head. Since v is a base point of a loop without entrances in X, $v \in \mathrm{Orb}^+(d(e_\lambda))$ implies that $d(e_\lambda) \in \mathrm{Orb}^+(v)$. Hence we have $r(e_\lambda) \in \mathrm{Orb}^+(v)$. This implies that $w \in \overline{\mathrm{Orb}^+(v)}$. Therefore we have $X^0 = \overline{\mathrm{Orb}^+(v)}$ for $v \in \mathrm{Per}_n(E)$. Thus $X^0 \in \mathcal{M}_{\mathrm{per}}(E)$. □

Definition 11.4. For $v \in BV(E)$ and $X^0 \in \mathcal{M}_{\mathrm{aper}}(E)$, we define $P_v = I_{\rho_v}$ and $P_{X^0} = I_{\rho_{X^0}}$.

Proposition 11.5. *The ideal P_x for $x \in BV(E) \amalg \mathcal{M}_{\mathrm{aper}}(E)$ is a prime ideal. If an ideal I satisfies $\rho_I = \rho_x$ for $x \in BV(E) \amalg \mathcal{M}_{\mathrm{aper}}(E)$, then $I = P_x$.*

Proof. This follows from Proposition 11.1, Proposition 11.2 and Proposition 11.3. □

In order to list all the prime ideals, the only remaining thing to do is to investigate the prime ideals P with $\rho_P = (X^0, X^0_{\mathrm{sg}})$ for $X^0 \in \mathcal{M}_{\mathrm{per}}(E)$.

We have the surjection $\mathrm{Per}(E) \ni v \to \overline{\mathrm{Orb}^+(v)} \in \mathcal{M}_{\mathrm{per}}(E)$. We first see this surjection carefully. For $v \in E^0$, let us denote by $[v] \subset E^0$ the equivalence class of v with respect to the equivalence relation on E^0 defined so that v and v' are equivalent if and only if $v' \in \mathrm{Orb}^+(v)$ and $v \in \mathrm{Orb}^+(v')$, which is equivalent to $\mathrm{Orb}^+(v) = \mathrm{Orb}^+(v')$. Thus $[v]$ is the union of $\{v\}$ and the set of vertices which lie on a loop whose base point is v.

Lemma 11.6. *For $v, v' \in \mathrm{Per}(E)$, $\overline{\mathrm{Orb}^+(v)} = \overline{\mathrm{Orb}^+(v')}$ if and only if $[v] = [v']$.*

Proof. It is clear that if $[v] = [v']$ then $\overline{\mathrm{Orb}^+(v)} = \overline{\mathrm{Orb}^+(v')}$. Conversely, suppose that $v, v' \in \mathrm{Per}(E)$ satisfy $\overline{\mathrm{Orb}^+(v)} = \overline{\mathrm{Orb}^+(v')}$. By definition, $\{v\}$ is open in $\overline{\mathrm{Orb}^+(v)}$. Hence $\{v\}$ is open in $\overline{\mathrm{Orb}^+(v')}$. This implies $v \in \mathrm{Orb}^+(v')$. Similarly we have $v' \in \mathrm{Orb}^+(v)$. Thus $[v] = [v']$. □

For $v \in \mathrm{Per}(E)$, we have $[v] = \{d(l_1), d(l_2), \ldots, d(l_n)\}$ where $l = (l_1, l_2, \ldots, l_n)$ is the unique simple loop whose base point is v. Hence the subset $\mathrm{Per}(E) \subset E^0$ is closed under the equivalence relation above. Let $[\mathrm{Per}(E)]$ be the set of equivalence classes in $\mathrm{Per}(E)$ of the equivalence relation above. For a positive integer n, the map $\mathrm{Per}_n(E) \ni v \mapsto [v] \in [\mathrm{Per}(E)]$ is $n:1$. By the lemma above, we get the following.

Corollary 11.7. *The map $[\mathrm{Per}(E)] \ni [v] \to \overline{\mathrm{Orb}^+(v)} \in \mathcal{M}_{\mathrm{per}}(E)$ is a well-defined bijection whose inverse is given by $\mathcal{M}_{\mathrm{per}}(E) \ni X^0 \to V \in [\mathrm{Per}(E)]$ where*

$$V = \{v \in \mathrm{Per}(E) \mid \overline{\mathrm{Orb}^+(v)} = X^0\}.$$

Take $X^0 \in \mathcal{M}_{\mathrm{per}}(E)$, and set $V = \{v \in \mathrm{Per}(E) \mid \overline{\mathrm{Orb}^+(v)} = X^0\} \in [\mathrm{Per}(E)]$. This subset V is the same as the one considered in the beginning of Section 8. Hence by Corollary 8.3, $S(V)$ is an open hereditary and saturated subset of X^0 contained in X^0_{rg}. Define a topological graph $X = (X^0, X^1, d_X, r_X)$ so that $X^1 = d^{-1}(X^0)$ and d_X, r_X are the restrictions of d, r. We can define $T^0\colon C_0(X^0) \to \mathcal{O}(E)/I_{\rho_{X^0}}$ and $T^1\colon C_{d_X}(X^1) \to \mathcal{O}(E)/I_{\rho_{X^0}}$ by $T^0(f|_{X^0}) = \omega(t^0(f))$ and $T^1(\xi|_{X^1}) = \omega(t^1(\xi))$ for $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$ where $\omega\colon \mathcal{O}(E) \to \mathcal{O}(E)/I_{\rho_{X^0}}$ is the natural quotient map. By the proof of Proposition 3.15, the pair $T = (T^0, T^1)$ induces the isomorphism $\mathcal{O}(X) \to \mathcal{O}(E)/I_{\rho_{X^0}}$. Let us define a subgraph $F = (F^0, F^1, d|_{F^1}, r|_{F^1})$ of X by $F^0 = S(V)$, $F^1 = (r_X)^{-1}(F^0)$. This subgraph F is nothing but the discrete graph considered in Proposition 8.5. Thus characteristic functions δ_v, δ_e of $v \in F^0$ and $e \in F^1$ are in $C_0(X^0)$ and $C_{d_X}(X^1)$, respectively.

Choose $v_0 \in V$, and let $l = (l_1, l_2, \ldots, l_n)$ be the unique simple loop whose base point is v_0. Thus we have $X^0 = \overline{\mathrm{Orb}^+(v_0)}$ and $V = \{d(l_1), d(l_2), \ldots, d(l_n)\}$. Let us set $p_0 = T^0(\delta_{v_0}) \in \mathcal{O}(E)/I_{\rho_{X^0}}$ and

$$u_0 = T^1(\delta_{l_1})T^1(\delta_{l_2}) \cdots T^1(\delta_{l_n}) \in \mathcal{O}(E)/I_{\rho_{X^0}}.$$

Note that the element u_0 can also be expressed as $T^n(\delta_l)$ for using $\delta_l \in C_{d_X}(X^n)$, and that we have $u_0^* u_0 = u_0 u_0^* = p_0$ (see the proof of Proposition 11.12).

Definition 11.8. For each $w \in \mathbb{T}$, we define an ideal $P_{V,w}$ of $\mathcal{O}(E)$ such that $P_{V,w}$ is the inverse image of the ideal generated by $u_0 - wp_0 \in \mathcal{O}(E)/I_{\rho_{X^0}}$ by the natural quotient map $\mathcal{O}(E) \to \mathcal{O}(E)/I_{\rho_{X^0}}$.

Although p_0, u_0 depend on the choice of $v_0 \in V$, the ideal $P_{V,w}$ does not depend. In fact, for $k = 1, 2, \ldots, n$ the elements p_k and u_k defined from $d(l_k) \in V$ as above satisfy

$$p_k = v_k^* p_0 v_k, \quad u_k = v_k^* u_0 v_k, \quad p_0 = v_k p_k v_k^*, \quad \text{and} \quad u_0 = v_k u_k v_k^*$$

for $v_k = T^1(\delta_{l_1})T^1(\delta_{l_2}) \cdots T^1(\delta_{l_k}) \in \mathcal{O}(E)/I_{\rho_{X^0}}$. This justifies the notation $P_{V,w}$.

Proposition 11.9. *For $z, w \in \mathbb{T}$, we have $\beta_z(P_{V,w}) = P_{V,z^{-n}w}$.*

Proof. This follows from the fact $\beta'_z(p_0) = p_0$ and $\beta'_z(u_0) = z^n u_0$ for $z \in \mathbb{T}$ where the action $\beta'\colon \mathbb{T} \curvearrowright \mathcal{O}(E)/I_{\rho_{X^0}}$ is induced by the gauge action β of $\mathcal{O}(X)$. □

We are going to show that $\{P_{V,w}\}_{w\in\mathbb{T}}$ is the list of all prime ideals P with $\rho_P = \rho_{X^0}$. In the next section, we will express the ideal $P_{V,1}$ as a kernel of a certain irreducible representation. We define an admissible pair ρ'_{X^0} of E by $\rho'_{X^0} = (X^0 \setminus S(V), X^0_{\mathrm{sg}})$. Then we have $\rho'_{X^0} \subset \rho_{X^0} = (X^0, X^0_{\mathrm{sg}})$.

Lemma 11.10. *For an ideal I of $\mathcal{O}(E)$, $\rho_I = \rho_{X^0}$ if and only if $I_{\rho_{X^0}} \subset I$ and $I_{\rho'_{X^0}} \not\subset I$.*

Proof. Since $I_{\rho_{X^0}} \subset I$ if and only if $\rho_I \subset \rho_{X^0}$, and $I_{\rho'_{X^0}} \not\subset I$ if and only if $\rho_I \not\subset \rho'_{X^0}$, it suffices to show that $\rho_I = \rho_{X^0}$ is equivalent to $\rho_I \subset \rho_{X^0}$ and $\rho_I \not\subset \rho'_{X^0}$ for an ideal I. Suppose that the pair $\rho_I = (X^0_I, Z_I)$ satisfies $\rho_I \subset \rho_{X^0} = (X^0, X^0_{\mathrm{sg}})$ and $\rho_I \not\subset \rho'_{X^0} = (X^0 \setminus S(V), X^0_{\mathrm{sg}})$. We have $S(V) \cap X^0_I \neq \emptyset$. This implies $V \cap X^0_I \neq \emptyset$ by Lemma 5.7. Thus $X^0 = \overline{\mathrm{Orb}^+(v_0)} \subset X^0_I \subset X^0$. Hence $X^0_I = X^0$. This implies $X^0_{\mathrm{sg}} = (X^0_I)_{\mathrm{sg}} \subset Z_I \subset X^0_{\mathrm{sg}}$. Thus $Z_I = X^0_{\mathrm{sg}}$. We have shown that $\rho_I = \rho_{X^0}$ when $\rho_I = (X^0_I, Z_I)$ satisfies $\rho_I \subset \rho_{X^0} = (X^0, X^0_{\mathrm{sg}})$ and $\rho_I \not\subset \rho'_{X^0} = (X^0 \setminus S(V), X^0_{\mathrm{sg}})$. The converse is clear because $\rho_{X^0} \not\subset \rho'_{X^0}$. □

Lemma 11.11. *The map $P \mapsto (P \cap I_{\rho'_{X^0}})/I_{\rho_{X^0}}$ is a bijection from the set of prime ideals P of $\mathcal{O}(E)$ with $\rho_P = \rho_{X^0}$ to the set of prime ideals of $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$.*

Proof. It is well-known and routine to check that the map $P \mapsto (P \cap I_{\rho'_{X^0}})/I_{\rho_{X^0}}$ is a bijection from the set of prime ideals P of $\mathcal{O}(E)$ with $I_{\rho_{X^0}} \subset P$ and $I_{\rho'_{X^0}} \not\subset P$ to the set of prime ideals of $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$ (see [RW, Proposition A.27] for the analogous statement for primitive ideals). Thus the conclusion follows from Lemma 11.10. □

Lemma 11.12. *The C^*-subalgebra $C^*(u_0)$ generated by u_0 is a hereditary and full subalgebra of the ideal $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$.*

Proof. If we identify $\mathcal{O}(X)$ and $\mathcal{O}(E)/I_{\rho_{X^0}}$ by the isomorphism induced by the pair $T = (T^0, T^1)$, then the admissible pair of X corresponding to the ideal $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$ of $\mathcal{O}(E)/I_{\rho_{X^0}}$ is $(X^0 \setminus S(V), X^0_{\mathrm{sg}})$. Since $X^0_{\mathrm{sg}} \cap S(V) = \emptyset$, Proposition 6.1 shows that $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$ is the ideal generated by $T^0(C_0(F^0)) \subset \mathcal{O}(E)/I_{\rho_{X^0}}$. By Proposition 6.3, the C^*-subalgebra A of $\mathcal{O}(E)/I_{\rho_{X^0}}$ generated by $T^0(C_0(F^0))$ and $T^1(C_d(F^1))$ is a hereditary and full subalgebra of the ideal $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$, and is naturally isomorphic to $\mathcal{O}(F)$. By Proposition 8.7 and its proof, there exists an isomorphism from $A \cong \mathcal{O}(F)$ to $C(\mathbb{T}) \otimes K(\ell^2(F^\infty))$ which sends p_0 and u_0 to $1 \otimes e_{l^\infty,l^\infty}$ and $w \otimes e_{l^\infty,l^\infty}$ respectively, where e_{l^∞,l^∞} is a minimal projection of $K(\ell^2(F^\infty))$ and w is the generating unitary of $C(\mathbb{T})$. Thus the C^*-subalgebra $C^*(u_0)$ of A generated by u_0 is sent onto $C(\mathbb{T}) \otimes e_{l^\infty,l^\infty}$ by the isomorphism $A \cong C(\mathbb{T}) \otimes K(\ell^2(F^\infty))$ described in the proof of Proposition 8.7. Hence $C^*(u_0)$ is a hereditary and full subalgebra of A, and hence of $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$. □

From this lemma, we see that $P_{V,w} \subset I_{\rho'_{X^0}}$ holds for all $w \in \mathbb{T}$. By Lemma 11.12, the map $P \mapsto P \cap C^*(u_0)$ is a bijection from the set of prime ideals P of $I_{\rho'_{X^0}}/I_{\rho_{X^0}}$ to the set of prime ideals of $C^*(u_0)$. As seen in the proof of Lemma 11.12, the C^*-algebra $C^*(u_0)$ is isomorphic to $C(\mathbb{T})$, and hence the set of prime ideals of $C^*(u_0)$ are $\{P_w\}_{w\in\mathbb{T}}$ where $P_w \subset C^*(u_0)$ is the ideal of $C^*(u_0)$ generated by $u_0 - wp_0 \in C^*(u_0)$. Combining these facts with Lemma 11.11, we get the following.

Proposition 11.13. *Let $X^0 \in \mathcal{M}_{\mathrm{per}}(E)$, and set $V = \{v \in \mathrm{Per}(E) \mid \overline{\mathrm{Orb}^+(v)} = X^0\}$. Then the set of all prime ideals P of $\mathcal{O}(E)$ satisfying $\rho_P = \rho_{X^0}$ is parameterized by $\mathbb{T}$ as $\{P_{V,w}\}_{w\in\mathbb{T}}$ such that $u_0 - wp_0 \in P_{V,w}/I_{\rho_{X^0}}$ for $w \in \mathbb{T}$.*

From the analysis above, we get the following theorem.

Theorem 11.14. *The set of all prime ideals of $\mathcal{O}(E)$ is the union of the following three disjoint sets;*

(i) $\{P_v \mid v \in BV(E)\}$,
(ii) $\{P_{X^0} \mid X^0 \in \mathcal{M}_{\rm aper}(E)\}$,
(iii) $\{P_{V,w} \mid V \in [\mathrm{Per}(E)], w \in \mathbb{T}\}$.

The prime ideals in (i) *and* (ii) *are gauge-invariant, and for $v \in \mathrm{Per}_n(E) \subset \mathrm{Per}(E)$ and $w \in \mathbb{T}$ we have $\{z \in \mathbb{T} \mid \beta_z(P_{[v],w}) = P_{[v],w}\} = \{z \in \mathbb{T} \mid z^n = 1\}$.*

12. Irreducible representations and primitive ideals

An ideal of a C^*-algebra is said to be *primitive* if it is a kernel of some irreducible representation. Every primitive ideal is prime ([RW, Proposition A.17 (b)]), and the converse is true when the C^*-algebra is separable ([RW, Proposition A.49]). In this section, we try to list all primitive ideals of $\mathcal{O}(E)$.

We define a subset $\mathcal{M}'(E) \subset \mathcal{M}(E)$ by

$$\mathcal{M}'(E) = \big\{\overline{\mathrm{Orb}(v,e)} \ \big|\ v \in E^0 \text{ and } e \text{ is a negative orbit of } v\big\}.$$

Then we have $\mathcal{M}_{\rm per}(E) \subset \mathcal{M}'(E)$. We define $\mathcal{M}'_{\rm aper}(E) \subset \mathcal{M}_{\rm aper}(E)$ by

$$\mathcal{M}'_{\rm aper}(E) = \mathcal{M}'(E) \cap \mathcal{M}_{\rm aper}(E) = \mathcal{M}'(E) \setminus \mathcal{M}_{\rm per}(E).$$

The following is the main theorem of this section.

Theorem 12.1. *The following ideals of $\mathcal{O}(E)$ are primitive;*

(i) $\{P_v \mid v \in BV(E)\}$,
(ii)' $\{P_{X^0} \mid X^0 \in \mathcal{M}'_{\rm aper}(E)\}$,
(iii) $\{P_{V,w} \mid V \in [\mathrm{Per}(E)], w \in \mathbb{T}\}$.

Remark 12.2. The author was not able to determine all primitive ideals. To determine all primitive ideals, it suffices to determine the subset $\widetilde{\mathcal{M}}_{\rm aper}(E) \subset \mathcal{M}_{\rm aper}(E)$ defined by

$$\widetilde{\mathcal{M}}_{\rm aper}(E) = \big\{X^0 \in \mathcal{M}_{\rm aper}(E) \ \big|\ P_{X^0} \text{ is a primitive ideal}\big\}$$

by Theorem 11.14 and Theorem 12.1. The theorem above implies that we have $\mathcal{M}'_{\rm aper}(E) \subset \widetilde{\mathcal{M}}_{\rm aper}(E) \subset \mathcal{M}(E)$. In Section 13, we will see that these two inclusions can be proper. The author does not know how to describe $\widetilde{\mathcal{M}}_{\rm aper}(E)$ in terms of the topological graph E.

The C^*-algebra $\mathcal{O}(E)$ is separable if and only if both E^0 and E^1 are second countable ([K1, Proposition 6.3]). Hence in this case every prime ideal of $\mathcal{O}(E)$ is primitive. The following generalizes this fact slightly.

Corollary 12.3. *When E^0 is second countable, every prime ideal of $\mathcal{O}(E)$ is primitive.*

Proof. By Lemma 4.14, we have $\mathcal{M}'_{\rm aper}(E) = \mathcal{M}_{\rm aper}(E)$ when E^0 is second countable. Hence the conclusion follows from Theorem 11.14 and Theorem 12.1. □

To prove Theorem 12.1, we need the following lemma.

Lemma 12.4. *For $X^0 \in \mathcal{M}'(E)$, either $X^0 = \overline{\mathrm{Orb}(r(l), l)}$ for an infinite path $l \in E^\infty$, or $X^0 = \overline{\mathrm{Orb}^+(v_0)}$ for $v_0 \in E^0_{\rm sg} \setminus BV(E)$.*

Proof. Take $v \in E^0$ and a negative orbit e of v such that $X^0 = \overline{\mathrm{Orb}(v,e)}$. When $e \in E^\infty$ we need to do nothing. Suppose $e \in E^*$. Then we have $X^0 = \overline{\mathrm{Orb}^+(v_0)}$ where $v_0 = d(e) \in E^0_{\mathrm{sg}}$. When $v_0 \notin BV(E)$, we are done. When $v_0 \in BV(E)$ then $v_0 \in X^0_{\mathrm{rg}}$. Hence we can find $l_1 \in d^{-1}(X^0)$ with $r(l_1) = v_0$ by Lemma 1.4. Set $v_1 = d(l_1) \in X^0$. Then we have $X^0 = \overline{\mathrm{Orb}^+(v_1)}$. When $v_1 \in E^0_{\mathrm{sg}} \setminus BV(E)$, we are done. Otherwise, we have $v_1 \in X^0_{\mathrm{rg}}$. Hence we can find $l_2 \in d^{-1}(X^0)$ with $r(l_2) = v_1$. Then we have $X^0 = \overline{\mathrm{Orb}^+(v_2)}$ where $v_2 = d(l_2) \in X^0$. By repeating this argument, either we can find $v_n \in E^0_{\mathrm{sg}} \setminus BV(E)$ with $X^0 = \overline{\mathrm{Orb}^+(v_n)}$ or we get $l = (l_1, l_2, \ldots) \in E^\infty$ such that $X^0 = \overline{\mathrm{Orb}^+(r(l), l)}$. We are done. □

By this lemma, Theorem 12.1 follows from the next two propositions.

Proposition 12.5. *For $v_0 \in E^0_{\mathrm{sg}}$, there exists an irreducible representation $\psi_{v_0} \colon \mathcal{O}(E) \to B(H_{v_0})$ such that $\rho_{\ker \psi_{v_0}} = (X^0, X^0_{\mathrm{sg}} \cup \{v_0\})$ where $X^0 = \overline{\mathrm{Orb}^+(v_0)}$.*

Proposition 12.6. *For $l \in E^\infty$, there exists an irreducible representation $\psi_l \colon \mathcal{O}(E) \to B(H_l)$ such that $\rho_{\ker \psi_l} = (X^0, X^0_{\mathrm{sg}})$ where $X^0 = \overline{\mathrm{Orb}(r(l), l)}$.*

Proof of Theorem 12.1. For $v_0 \in BV(E) \subset E^0_{\mathrm{sg}}$, the ideal $\ker \psi_{v_0}$ in Proposition 12.5 coincides with P_{v_0} by Proposition 11.5. Hence P_{v_0} is primitive.

For $v_0 \in E^0_{\mathrm{sg}} \setminus BV(E)$, we have $(X^0, X^0_{\mathrm{sg}} \cup \{v_0\}) = \rho_{X^0}$ where $X^0 = \overline{\mathrm{Orb}^+(v_0)}$. Hence in a similar way as above using Proposition 11.5 with the help of Lemma 12.4, Proposition 12.5 and Proposition 12.6, we can prove that P_{X^0} is primitive for $X^0 \in \mathcal{M}'_{\mathrm{aper}}(E)$.

Let us take $v_0 \in \mathrm{Per}_n(E) \subset \mathrm{Per}(E)$, and set $X^0 = \overline{\mathrm{Orb}^+(v_0)}$ and $V = [v_0]$. Let $l \in E^\infty$ be the unique negative orbit of v_0. Then we have $X^0 = \overline{\mathrm{Orb}(r(l), l)}$. Hence by Proposition 12.6, we get a primitive ideal P with $\rho_P = \rho_{X^0}$. Since P is prime, $P = P_{V,w_0}$ for some $w_0 \in \mathbb{T}$ by Proposition 11.13. For any $w \in \mathbb{T}$, $P_{V,w} = \beta_z(P)$ with $z \in \mathbb{T}$ satisfying $w = z^{-n} w_0$. Hence $P_{V,w}$ is primitive for all $w \in \mathbb{T}$. This completes the proof of Theorem 12.1 modulo the proofs of Proposition 12.5 and Proposition 12.6. □

We will prove Proposition 12.5 and Proposition 12.6 to finish the proof of Theorem 12.1.

Take $v_0 \in E^0_{\mathrm{sg}}$. We define a set Λ_{v_0} by $\Lambda_{v_0} = \{\lambda \in E^* \mid d(\lambda) = v_0\}$. For $\lambda \in \Lambda_{v_0}$ with $\lambda \in E^k$, we set its length by $|\lambda| = k$. We define $E^1{}_d\!\times_r \Lambda_{v_0}$ by

$$E^1{}_d\!\times_r \Lambda_{v_0} = \big\{(e, \lambda) \in E^1 \times \Lambda_{v_0} \;\big|\; d(e) = r(\lambda)\big\}.$$

For $(e, \lambda) \in E^1{}_d\!\times_r \Lambda_{v_0}$, we define $e\lambda \in \Lambda_{v_0}$ by $ev_0 = e$ and $e\lambda = (e, e_1, \ldots, e_k)$ for $\lambda = (e_1, \ldots, e_k) \in \Lambda_{v_0}$. Then we have $|e\lambda| = |\lambda| + 1$ and the map

$$E^1{}_d\!\times_r \Lambda_{v_0} \ni (e, \lambda) \mapsto e\lambda \in \Lambda_{v_0} \setminus \{v_0\}$$

is a bijection. Let H_{v_0} be the Hilbert space whose complete orthonormal system is given by $\{\delta_\lambda\}_{\lambda \in \Lambda_{v_0}}$.

Definition 12.7. We define a $*$-homomorphism $T^0_{v_0} \colon C_0(E^0) \to B(H_{v_0})$ and a linear map $T^1_{v_0} \colon C_d(E^1) \to B(H_{v_0})$ by

$$T^0_{v_0}(f)\delta_\lambda = f(r(\lambda))\delta_\lambda, \quad T^1_{v_0}(\xi)\delta_\lambda = \sum_{e \in d^{-1}(r(\lambda))} \xi(e)\delta_{e\lambda}$$

for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $\lambda \in \Lambda_{v_0}$.

It is not difficult to see that $T^1_{v_0}$ is a well defined norm-decreasing linear map. We will show that $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ is a Cuntz-Krieger E-pair.

Lemma 12.8. *For $\xi \in C_d(E^1)$ and $(e,\lambda) \in E^1{}_d\times_r \Lambda_{v_0}$, we have*

$$T^1_{v_0}(\xi)^*\delta_{v_0} = 0, \quad T^1_{v_0}(\xi)^*\delta_{e\lambda} = \overline{\xi(e)}\delta_\lambda.$$

Proof. Straightforward. □

Lemma 12.9. *For $\xi, \eta \in C_d(E^1)$, we have $T^1_{v_0}(\xi)^*T^1_{v_0}(\eta) = T^0_{v_0}(\langle \xi, \eta\rangle)$.*

Proof. For $\lambda \in \Lambda_{v_0}$, we have

$$\begin{aligned} T^1_{v_0}(\xi)^*T^1_{v_0}(\eta)\delta_\lambda &= T^1_{v_0}(\xi)^*\bigg(\sum_{e\in d^{-1}(r(\lambda))}\eta(e)\delta_{e\lambda}\bigg) \\ &= \sum_{e\in d^{-1}(r(\lambda))}\overline{\xi(e)}\eta(e)\delta_\lambda \\ &= \langle \xi, \eta\rangle(r(\lambda))\delta_\lambda \\ &= T^0_{v_0}(\langle \xi, \eta\rangle)\delta_\lambda. \end{aligned}$$

This shows $T^1_{v_0}(\xi)^*T^1_{v_0}(\eta) = T^0_{v_0}(\langle \xi, \eta\rangle)$. □

Lemma 12.10. *For $f \in C_0(E^0)$ and $\xi \in C_d(E^1)$, we have $T^0_{v_0}(f)T^1_{v_0}(\xi) = T^1_{v_0}(\pi_r(f)\xi)$.*

Proof. This follows from the computation

$$\begin{aligned} T^0_{v_0}(f)T^1_{v_0}(\xi)\delta_\lambda &= T^0_{v_0}(f)\bigg(\sum_{e\in d^{-1}(r(\lambda))}\xi(e)\delta_{e\lambda}\bigg) \\ &= \sum_{e\in d^{-1}(r(\lambda))} f(r(e\lambda))\xi(e)\delta_{e\lambda} \\ &= \sum_{e\in d^{-1}(r(\lambda))} (\pi_r(f)\xi)(e)\delta_{e\lambda} \\ &= T^1_{v_0}(\pi_r(f)\xi)\delta_\lambda, \end{aligned}$$

for $\lambda \in \Lambda_{v_0}$. □

By Lemma 12.9 and Lemma 12.10, the pair $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ is a Toeplitz E-pair. We will show that it is a Cuntz-Krieger E-pair. Let $\Phi_{v_0}\colon \mathcal{K}(C_d(E^1)) \to B(H_{v_0})$ be the $*$-homomorphism defined by $\Phi_{v_0}(\theta_{\xi,\eta}) = T^1_{v_0}(\xi)T^1_{v_0}(\eta)^*$ for $\xi, \eta \in C_d(E^1)$. Recall that the left action $\pi_r\colon C_0(E^0) \to \mathcal{L}(C_d(E^1))$ is defined by $\pi_r(f) = \pi(f\circ r)$ for $f \in C_0(E^0)$, where $\pi\colon C_0(E^1) \to \mathcal{K}(C_d(E^1))$ is defined by $(\pi(F)\xi)(e) = F(e)\xi(e)$ for $F \in C_0(E^1)$, $\xi \in C_d(E^1)$ and $e \in E^1$.

Lemma 12.11. *For $F \in C_0(E^1)$ and $(e,\lambda) \in E^1{}_d\times_r \Lambda_{v_0}$, we have*

$$\Phi_{v_0}(\pi(F))\delta_{v_0} = 0, \quad \Phi_{v_0}(\pi(F))\delta_{e\lambda} = F(e)\delta_{e\lambda}.$$

Proof. Let us take $\xi, \eta \in C_d(E^1)$ such that $\xi(e)\overline{\eta}(e') = 0$ for $e, e' \in E^1$ with $e \neq e'$ and $d(e) = d(e')$. We set $F = \xi\overline{\eta} \in C_0(E^1)$. Since the linear span of such F is dense in $C_0(E^1)$ by [K1, Lemma 1.16], it suffices to show the equalities for this F. We have $\pi(F) = \theta_{\xi,\eta}$ by [K1, Lemma 1.15]. Hence we have

$$\Phi_{v_0}(\pi(F))\delta_{v_0} = T^1_{v_0}(\xi)T^1_{v_0}(\eta)^*\delta_{v_0} = 0,$$

and

$$\begin{aligned}
\Phi_{v_0}(\pi(F))\delta_{e\lambda} &= T^1_{v_0}(\xi)T^1_{v_0}(\eta)^*\delta_{e\lambda} \\
&= T^1_{v_0}(\xi)\overline{\eta(e)}\delta_\lambda \\
&= \sum_{e'\in d^{-1}(r(\lambda))} \xi(e')\overline{\eta(e)}\delta_{e'\lambda} \\
&= \xi(e)\overline{\eta(e)}\delta_{e\lambda} \\
&= F(e)\delta_{e\lambda}.
\end{aligned}$$

for $(e,\lambda) \in E^1{}_d\times_r \Lambda_{v_0}$. We are done. □

Proposition 12.12. *The pair $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ is a Cuntz-Krieger E-pair.*

Proof. By Lemma 12.9 and Lemma 12.10, T_{v_0} is a Toeplitz E-pair. Take $f \in C_0(E^0_{\mathrm{rg}})$. We have $T^0_{v_0}(f)\delta_{v_0} = f(v_0)\delta_{v_0} = 0$ because $v_0 \in E^0_{\mathrm{sg}}$. We also have

$$\Phi_{v_0}(\pi_r(f))\delta_{v_0} = \Phi_{v_0}(\pi(f\circ r))\delta_{v_0} = 0$$

by Lemma 12.11 because $f\circ r \in C_0(E^1)$. Lemma 12.11 also gives

$$\begin{aligned}
\Phi_{v_0}(\pi_r(f))\delta_{e\lambda} &= \Phi_{v_0}(\pi(f\circ r))\delta_{e\lambda} \\
&= f(r(e))\delta_{e\lambda} \\
&= T^0_{v_0}(f)\delta_{e\lambda}
\end{aligned}$$

for $(e,\lambda) \in E^1{}_d\times_r \Lambda_{v_0}$. Thus we have $\Phi_{v_0}(\pi_r(f)) = T^0_{v_0}(f)$. This shows that the pair $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ is a Cuntz-Krieger E-pair. □

By Proposition 12.12, we get a representation $\psi_{v_0}\colon \mathcal{O}(E) \to B(H_{v_0})$ such that $T^i_{v_0} = \psi_{v_0}\circ t^i$ for $i = 0, 1$.

Proposition 12.13. *The representation ψ_{v_0} is irreducible.*

Proof. We will prove that the weak closure of $\psi_{v_0}(\mathcal{O}(E))$ is $B(H_{v_0})$. For an approximate unit $\{f_\nu\}$ of $C_0(E^0)$, the net $\{T^0_{v_0}(f_\nu)\}$ converges to the identity $1 \in B(H_{v_0})$ weakly. For an approximate unit $\{F_\nu\}$ of $C_0(E^1)$, the net $\{\Phi_{v_0}(\pi(F_\nu))\}$ converges to the projection onto the orthogonal complement of $\mathbb{C}\delta_{v_0}$ weakly. Hence the rank-one projection $p_{v_0} \in B(H_{v_0})$ onto $\mathbb{C}\delta_{v_0}$ is in the weak closure of $\psi_{v_0}(\mathcal{O}(E))$. For $\lambda \in \Lambda_{v_0}$ with $|\lambda| = 1$, we can find $\xi \in C_d(E^1)$ with $\xi(\lambda) = 1$ and $\xi(e) = 0$ for $e \in d^{-1}(v_0)$ with $e \neq \lambda$. Then $T^1_{v_0}(\xi)\delta_{v_0} = \delta_\lambda$. Similarly, for each $(e,\lambda) \in E^1{}_d\times_r \Lambda_{v_0}$, we can find $\xi \in C_d(E^1)$ such that $T^1_{v_0}(\xi)\delta_\lambda = \delta_{e\lambda}$. Hence for every $\lambda \in \Lambda_{v_0}$ with $k = |\lambda| \geq 1$, there exists $x_\lambda = T^1_{v_0}(\xi_1)T^1_{v_0}(\xi_2)\dots T^1_{v_0}(\xi_k) \in \psi_{v_0}(\mathcal{O}(E))$ such that $x_\lambda\delta_{v_0} = \delta_\lambda$. Let us set $u_{v_0} = p_{v_0}$ and $u_\lambda = x_\lambda p_{v_0}$ for $\lambda \in \Lambda_{v_0}\setminus\{v_0\}$, which are in the weak closure of $\psi_{v_0}(\mathcal{O}(E))$. Since the von Neumann algebra generated by $\{u_\lambda\}_{\lambda\in\Lambda_{v_0}}$ is $B(H_{v_0})$, the weak closure of $\psi_{v_0}(\mathcal{O}(E))$ is $B(H_{v_0})$. We are done. □

Proof of Proposition 12.5. To finish the proof of Proposition 12.5, it remains to check $\rho_{\ker\psi_{v_0}} = (X^0, X^0_{\mathrm{sg}}\cup\{v_0\})$ where $X^0 = \overline{\mathrm{Orb}^+(v_0)}$.

From $r(\Lambda_{v_0}) = \mathrm{Orb}^+(v_0)$, we have $\ker T^0_{v_0} = C_0(E^0\setminus\overline{\mathrm{Orb}^+(v_0)}) = C_0(E^0\setminus X^0)$. This shows $X^0_{\ker\psi_{v_0}} = X^0$. By Proposition 2.8, we have $X^0_{\mathrm{sg}} \subset Z_{\ker\psi_{v_0}}$. By the same way to the proof of Lemma 12.11, we have $\Phi_{v_0}(k)\delta_{v_0} = 0$ for all $k \in \mathcal{K}(C_d(E^1))$. Hence $f(v_0) = 0$

for all $f \in C_0(E^0)$ with $T^0_{v_0}(f) \in \Phi_{v_0}\big(\mathcal{K}(C_d(E^1))\big)$. This shows $v_0 \in Z_{\ker \psi_{v_0}}$. To prove the other inclusion $Z_{\ker \psi_{v_0}} \subset X^0_{\mathrm{sg}} \cup \{v_0\}$, take

$$f \in C_0\big(E^0 \setminus (X^0_{\mathrm{sg}} \cup \{v_0\})\big),$$

and we will prove $T^0_{v_0}(f) \in \Phi_{v_0}\big(\mathcal{K}(C_d(E^1))\big)$. We have $(f \circ r)|_{X^1} \in C_0(X^1)$ because $f|_{X^0} \in C_0(X^0_{\mathrm{rg}})$. Hence there exists $F \in C_0(E^1)$ such that $F(e) = f(r(e))$ for $e \in X^1$. For $(e, \lambda) \in E^1{}_d\times_r \Lambda_{v_0}$, we have

$$T^0_{v_0}(f)\delta_{e\lambda} = f(r(e))\delta_{e\lambda} = F(e)\delta_{e\lambda} = \Phi_{v_0}(\pi(F))\delta_{e\lambda}$$

because $e \in X^1$. Since $f(v_0) = 0$, we have $T^0_{v_0}(f)\delta_{v_0} = 0 = \Phi_{v_0}(\pi(F))\delta_{v_0}$. Hence $T^0_{v_0}(f) = \Phi_{v_0}(\pi(F)) \in \Phi_{v_0}\big(\mathcal{K}(C_d(E^1))\big)$. Therefore we get $\rho_{\ker \psi_{v_0}} = (X^0, X^0_{\mathrm{sg}} \cup \{v_0\})$. □

Next we will prove Proposition 12.6. The method is almost identical to the proof of Proposition 12.5, hence we just sketch it. In fact, the proof of Proposition 12.6 is easier than the one of Proposition 12.5 because in the case of Proposition 12.5 there exists a special element $v_0 \in \Lambda_{v_0}$ which cannot be expressed as $e\lambda$, but no such elements appear in the case of Proposition 12.6.

Let us take $l = (l_1, l_2, \ldots, l_n, \ldots) \in E^\infty$. We denote by Λ_l the set of

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \in E^\infty$$

satisfying that there exist $k \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $\lambda_{n+k} = l_n$ for all $n \geq N$. We define $E^1{}_d\times_r \Lambda_l$, and the bijective map

$$E^1{}_d\times_r \Lambda_l \ni (e, \lambda) \mapsto e\lambda \in \Lambda_l$$

in a similar way as before. Let H_l be the Hilbert space whose complete orthonormal system is given by $\{\delta_\lambda\}_{\lambda\in\Lambda_l}$.

Definition 12.14. We define a $*$-homomorphism $T^0_l \colon C_0(E^0) \to B(H_l)$ and a linear map $T^1_l \colon C_d(E^1) \to B(H_l)$ by

$$T^0_l(f)\delta_\lambda = f(r(\lambda))\delta_\lambda, \quad T^1_l(\xi)\delta_\lambda = \sum_{e\in d^{-1}(r(\lambda))} \xi(e)\delta_{e\lambda}$$

for $f \in C_0(E^0)$, $\xi \in C_d(E^1)$ and $\lambda \in \Lambda_l$.

In a similar way to the proof of Proposition 12.12, we can show that $T_l = (T^0_l, T^1_l)$ is a Cuntz-Krieger E-pair. Thus we get a representation $\psi_l \colon \mathcal{O}(E) \to B(H_l)$ such that $T^i_l = \psi_l \circ t^i$ for $i = 0, 1$.

Proposition 12.15. *The representation ψ_l is irreducible.*

Proof. First note that we can define the set $E^n{}_d\times_r \Lambda_l$ and the bijective map

$$E^n{}_d\times_r \Lambda_l \ni (e, \lambda) \mapsto e\lambda \in \Lambda_l$$

for a positive integer n. For $\xi \in C_d(E^n)$ and $(e, \lambda) \in E^n{}_d\times_r \Lambda_l$, we have $T^n_l(\xi)\delta_{e\lambda} = \xi(e)\delta_\lambda$. Hence if $\xi \in C_d(E^n)$ satisfies $\xi(e) = 1$ and $\xi(e') = 0$ for $e' \in E^n$ with $d(e') = d(e)$ and $e' \neq e$, then $T^n_l(\xi)\delta_\lambda = \delta_{e\lambda}$ and $T^n_l(\xi)^*\delta_{e\lambda} = \delta_\lambda$. Hence for each $\lambda \in \Lambda_l$ we can find $x_\lambda \in \psi_l(\mathcal{O}(E))$ such that $x_\lambda\delta_l = \delta_\lambda$.

Let $p_l \in B(H_l)$ be the rank-one projection onto $\mathbb{C}\delta_l$. To prove that p_l is in the weak closure of $\psi_l(\mathcal{O}(E))$, it suffices to show that for each finite subset $Y \subset \Lambda_l \setminus \{l\}$ there exists

$x \in \psi_l(\mathcal{O}(E))$ such that $x\delta_l = \delta_l$ and $x\delta_\lambda = 0$ for $\lambda \in Y$. Take a finite subset $Y \subset \Lambda_l \setminus \{l\}$. For a positive integer n, we define $l_{[1,n]} = (l_1, l_2, \ldots, l_n) \in E^n$ and

$$Y_{[1,n]} = \{e \in E^n \mid (e, \lambda) \in {E^n}_d\times_r \Lambda_l \text{ with } e\lambda \in Y\}.$$

Since $l \notin Y$, we have $l_{[1,n]} \notin Y_{[1,n]}$ for a sufficiently large integer n. Choose $\xi \in C_d(E^n)$ such that $\xi(l_{[1,n]}) = 1$ and $\xi(e) = 0$ for every $e \in Y_{[1,n]}$. Then $x = T_l^n(\xi)T_l^n(\xi)^* \in \psi_l(\mathcal{O}(E))$ satisfies $x\delta_l = \delta_l$ and $x\delta_\lambda = 0$ for $\lambda \in Y$. This shows that p_l is in the weak closure of $\psi_l(\mathcal{O}(E))$. Since the von Neumann algebra generated by $\{x_\lambda p_l\}_{\lambda\in\Lambda_l}$ is $B(H_l)$, the weak closure of $\psi_l(\mathcal{O}(E))$ is $B(H_l)$. The proof is completed. $\square$

Proof of Proposition 12.6. To finish the proof of Proposition 12.6, it remains to check $\rho_{\ker\psi_l} = (X^0, X^0_{\mathrm{sg}})$ where $X^0 = \overline{\mathrm{Orb}(r(l), l)}$. From $r(\Lambda_l) = \mathrm{Orb}(r(l), l)$, we have $\ker T_l^0 = C_0(E^0 \setminus X^0)$. This shows $X^0_{\ker\psi_l} = X^0$. The proof of $Z_{\ker\psi_l} = X^0_{\mathrm{sg}}$ is the same as the proof of Proposition 12.5. $\square$

Thus we have completed the proof of Theorem 12.1. In the proof, we used Proposition 11.5 which depends heavily on the Cuntz-Krieger Uniqueness Theorem (Proposition 6.7). In the following, we give the direct proof of the gauge-invariance of the kernels of the irreducible representations we consider, so that we can use the Gauge Invariant Uniqueness Theorem ([K1, Theorem 4.5]) instead of Proposition 11.5. Note that the proof of the Gauge Invariant Uniqueness Theorem is much shorter and easier than the one of the Cuntz-Krieger Uniqueness Theorem. We also analyze the primitive ideal which is not gauge-invariant in the detail.

Lemma 12.16. *For $v_0 \in E^0_{\mathrm{sg}}$, the primitive ideal $\ker\psi_{v_0}$ is gauge-invariant.*

Proof. It suffices to see that the Cuntz-Krieger E-pair $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ admits a gauge action. For $z \in \mathbb{T}$, we define a unitary $u_z \in B(H_{v_0})$ by $u_z\delta_\lambda = z^{|\lambda|}\delta_\lambda$ for $\lambda \in \Lambda_{v_0}$. Then it is easy to see that the automorphism $\mathrm{Ad}(u_z)$ of $B(H_{v_0})$ defined by $\mathrm{Ad}(u_z)(x) = u_z x u_z^*$ for $z \in \mathbb{T}$ is a gauge action for T_{v_0}. We are done. $\square$

Definition 12.17. An infinite path $l = (l_1, l_2, \ldots, l_n, \ldots) \in E^\infty$ is said to be *periodic* if there exist positive integers k and N such that $l_{n+k} = l_n$ for all $n \geq N$. Otherwise $l \in E^\infty$ is said to be *aperiodic*.

Lemma 12.18. *When $l \in E^\infty$ is aperiodic, the primitive ideal $\ker\psi_l$ is gauge-invariant.*

Proof. It suffices to see that the Cuntz-Krieger E-pair $T_l = (T_l^0, T_l^1)$ admits a gauge action. When l is aperiodic, for each $\lambda \in \Lambda_l$, an integer $k \in \mathbb{Z}$ satisfying $\lambda_{n+k} = l_n$ for large n is unique. Hence we can write $c_\lambda = k \in \mathbb{Z}$. It is easy to see that $c_{e\lambda} = c_\lambda + 1$ for $(e, \lambda) \in {E^1}_d\times_r \Lambda_l$. Now we define a unitary $u_z \in B(H_l)$ for $z \in \mathbb{T}$ by $u_z\delta_\lambda = z^{c_\lambda}\delta_\lambda$ for $\lambda \in \Lambda_l$. Then it is easy to see that the automorphism $\mathrm{Ad}(u_z)$ of $B(H_l)$ defined by $\mathrm{Ad}(u_z)(x) = u_z x u_z^*$ for $z \in \mathbb{T}$ is a gauge action for T_l. We are done. $\square$

Let l be a periodic infinite path. Then there exists a simple loop $l' = (e_1, e_2, \ldots, e_n)$ such that $l = (e', l', l', \ldots, l', \ldots)$ for some $e' \in E^*$. Set $v_0 = r(l') \in E^0$. Then the closed set $X^0 = \overline{\mathrm{Orb}(r(l), l)}$ coincides with $\overline{\mathrm{Orb}^+(v_0)}$. Let us define $U = \{e_1, e_2, \ldots, e_n\} \subset E^1$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \in \Lambda_l$, we define $|\lambda| \in \mathbb{N}$ by $|\lambda| = 0$ if $\lambda_k \in U$ for all k, and $|\lambda| = \max\{k \mid \lambda_k \notin U\}$ otherwise. For $(e, \lambda) \in {E^1}_d\times_r \Lambda_l$, we have $|e\lambda| = 0$ if and only if $e \in U$ and $|\lambda| = 0$. Otherwise, we have $|e\lambda| = |\lambda| + 1$. Note that the number of elements $\lambda \in \Lambda_l$ with $|\lambda| = 0$ is n.

Let $\psi_l'\colon \mathcal{O}(E)\to B(H_l)/K(H_l)$ be the composition of the representation $\psi_l\colon \mathcal{O}(E)\to B(H_l)$ and the natural surjection $\Pi\colon B(H_l)\to B(H_l)/K(H_l)$. We have $\ker\psi_l\subset\ker\psi_l'$.

Proposition 12.19. *The ideal* $\ker\psi_l'$ *is gauge-invariant.*

Proof. It suffices to see that $(\Pi\circ T_l^0, \Pi\circ T_l^1)$ admits a gauge action. For each $z\in\mathbb{T}$, define a unitary $u_z\in B(H_l)$ by $u_z\delta_\lambda = z^{|\lambda|}\delta_\lambda$ for $\lambda\in\Lambda_l$. Then we have $u_zT_l^0(f)u_z^* = T_l^0(f)$ for all $f\in C_0(E^0)$. For $\xi\in C_d(E^1)$, we have $u_zT_l^1(\xi)u_z^* = zT_l^1(\xi)$ on the subspace generated by $\{\delta_\lambda\}_{|\lambda|\geq 1}$ which is codimension finite. Hence if we define an automorphism β_z' of $B(H_l)/K(H_l)$ by $\beta_z'(x) = \Pi(u_z)x\Pi(u_z)^*$ for $z\in\mathbb{T}$, then β' is a gauge action for $(\Pi\circ T_l^0, \Pi\circ T_l^1)$. This completes the proof. □

Lemma 12.20. *We have* $X^0_{\ker\psi_l'}\subset X^0$, *and* $v\in X^0\setminus X^0_{\ker\psi_l'}$ *if and only if* $\{v\}$ *is open in* X^0 *and the set* $\{\lambda\in\Lambda_l\mid r(\lambda)=v\}$ *is finite.*

Proof. Since $\ker\psi_l\subset\ker\psi_l'$, we have $X^0_{\ker\psi_l'}\subset X^0_{\ker\psi_l} = X^0$. Take $v\in X^0\setminus X^0_{\ker\psi_l'}$. Then there exists $f\in C_0(E^0)$ such that $t^0(f)\in\ker\psi_l'$ and $f(v)=1$. Set

$$V = \{v\in X^0\mid f(v)\geq 1/2\}$$

which is a neighborhood of $v\in X^0$. Since $t^0(f)\in\ker\psi_l'$, we have $T^0(f) = \psi_l(t^0(f))\in K(H_l)$. Hence the projection $p = \chi_{[1/2,\infty)}(T^0(f))$ is of finite rank. By the definition of T^0, p is the projection onto the subspace spanned by

$$\{\delta_\lambda\mid f(r(\lambda))\geq 1/2\} = \{\delta_\lambda\mid r(\lambda)\in V\}.$$

Thus $\{\lambda\in\Lambda_l\mid r(\lambda)\in V\}$ is finite. This shows that $\{\lambda\in\Lambda_l\mid r(\lambda)=v\}$ is finite. Since $\mathrm{Orb}^+(v_0) = \mathrm{Orb}(r(l),l)$ is the image of the map $r\colon\Lambda_l\to E^0$, $\mathrm{Orb}^+(v_0)\cap V$ is a finite set. Since $\mathrm{Orb}^+(v_0)$ is dense in X^0, V is a finite subset of $\mathrm{Orb}^+(v_0)$. This shows that $\{v\}$ is open.

Conversely suppose that $\{v\}$ is open in X^0 and the set $\{\lambda\in\Lambda_l\mid r(\lambda)=v\}$ is finite. We can find $f\in C_0(E^0)$ such that $f(v)=1$ and $f(v')=0$ for $v'\in X^0\setminus\{v\}$. Then $T^0(f)$ is a projection onto the subspace spanned by the finite set $\{\delta_\lambda\mid r(\lambda)=v\}$. Hence $t^0(f)\in\ker\psi_l'$. This shows $v\notin X^0_{\ker\psi_l'}$. We are done. □

Proposition 12.21. *When* $v_0\in\mathrm{Aper}(E)$, *we have* $\ker\psi_l = \ker\psi_l'$ *and hence the primitive ideal* $\ker\psi_l$ *is gauge-invariant.*

Proof. By Lemma 12.20, $v_0\in\mathrm{Aper}(E)$ implies $v_0\in X^0_{\ker\psi_l'}$. Hence

$$X^0 = \overline{\mathrm{Orb}^+(v_0)}\subset X^0_{\ker\psi_l'}\subset X^0.$$

This shows $X^0_{\ker\psi_l'} = X^0$. Hence $X^0_{\mathrm{sg}} = (X^0_{\ker\psi_l'})_{\mathrm{sg}}\subset Z_{\ker\psi_l'}$. Since $\ker\psi_l\subset\ker\psi_l'$, we have $Z_{\ker\psi_l'}\subset X^0_{\mathrm{sg}}$. Thus $Z_{\ker\psi_l'} = X^0_{\mathrm{sg}}$. We have shown that $\rho_{\ker\psi_l'} = \rho_{\ker\psi_l}$. By Proposition 12.19, we have $\ker\psi_l' = I_{\rho_{\ker\psi_l'}} = I_{\rho_{\ker\psi_l}}\subset\ker\psi_l$. Therefore we have $\ker\psi_l = \ker\psi_l'$. □

Proposition 12.22. *When* $v_0\in\mathrm{Per}(E)$, *we have* $\ker\psi_l = P_{V,1}$ *and hence the primitive ideal* $\ker\psi_l$ *is not gauge-invariant.*

Proof. We consider X^n as a closed subset of E^n. Since $v_0\in\mathrm{Per}(E)$, $\{l'\}$ is open in X^n. Hence there exists $\xi\in C_d(E^n)$ such that $\xi(l')=1$ and $\xi(e)=0$ for $e\in X^n\setminus\{l'\}$. Note that the image of $t^n(\xi)\in\mathcal{O}(E)$ in $\mathcal{O}(E)/I_{\rho_{X^0}}$ is the element u_0 defined in the previous

section (see the remarks before Definition 11.8). We can see that $T_l^n(\xi)\delta_\lambda = \delta_\lambda$ when $|\lambda| = 0$, and $T_l^n(\xi)\delta_\lambda = 0$ when $|\lambda| \geq 1$. Hence $T_l^n(\xi) \in B(H_l)$ is the projection onto the subspace spanned by $\{\delta_\lambda\}_{|\lambda|=0}$. This shows $t^n(\xi) - t^n(\xi)^* t^n(\xi) \in \ker\psi_l$. Thus $\ker\psi_l$ is a prime ideal with $\rho_{\ker\psi_l} = \rho_{X^0}$ and $u_0 - u_0^* u_0 \in \ker\psi_l / I_{\rho_{X^0}}$. Therefore by Proposition 11.13 we have $\ker\psi_l = P_{V,1}$. □

When $v_0 \in \operatorname{Per}(E)$, the admissible pair ρ'_{X^0} was defined by $\rho'_{X^0} = (X^0 \setminus S(V), X^0_{\mathrm{sg}})$ where $V = \{d(e_1), d(e_2), \ldots, d(e_n)\}$.

Proposition 12.23. *When $v_0 \in \operatorname{Per}(E)$, we have $\rho_{\ker\psi'_l} = \rho'_{X^0}$ and hence $\ker\psi'_l = I_{\rho'_{X^0}}$.*

Proof. By Proposition 8.2 and Lemma 12.20, we have $X^0 \setminus X^0_{\ker\psi'_l} = S(V)$. Hence $X^0_{\ker\psi'_l} = X^0 \setminus S(V)$. Since $\ker\psi_l \subset \ker\psi'_l$, we have $Z_{\ker\psi'_l} \subset Z_{\ker\psi_l} = X^0_{\mathrm{sg}}$. We will show the other inclusion $X^0_{\mathrm{sg}} \subset Z_{\ker\psi'_l}$. To do so, it suffices to show $T_l^0(f) \notin K(H_l) + \Phi_l\big(\mathcal{K}(C_d(E^1))\big)$ for $f \in C_0(E^0)$ with $f(v) \neq 0$ for some $v \in X^0_{\mathrm{sg}}$, where $\Phi_l\colon \mathcal{K}(C_d(E^1)) \to B(H_l)$ is defined by $\Phi_l(\theta_{\xi,\eta}) = T_l^1(\xi)T_l^1(\eta)^*$ for $\xi, \eta \in C_d(E^1)$. For $x \in K(H_l)$ and $\varepsilon > 0$,

$$\big\{e \in E^1 \;\big|\; \|x\delta_{e\lambda}\| > \varepsilon, (e,\lambda) \in E^1{}_d{\times_r}\, \Lambda_l\big\}$$

is finite. For $\xi, \eta \in C_d(E^1)$ and $(e,\lambda) \in E^1{}_d{\times_r}\, \Lambda_l$, we have

$$\begin{aligned}
\big\|\Phi_l(\theta_{\xi,\eta})\delta_{e\lambda}\big\| &= \big\|\big(T_l^1(\xi)T_l^1(\eta)^*\big)\delta_{e\lambda}\big\| \\
&= \big\|T_l^1(\xi)\big(\overline{\eta(e)}\delta_\lambda\big)\big\| \\
&= \big|\overline{\eta(e)}\big|\Bigg\|\sum_{e' \in d^{-1}(r(\lambda))} \xi(e')\delta_{e'\lambda}\Bigg\| \\
&= |\eta(e)|\Bigg(\sum_{e' \in d^{-1}(d(e))} \big|\xi(e')\big|^2\Bigg)^{1/2} \\
&= \big|(\eta f)(e)\big|
\end{aligned}$$

where $f = \langle\xi,\xi\rangle^{1/2} \in C_0(E^0)$. Hence for all $x \in \Phi_l\big(\mathcal{K}(C_d(E^1))\big)$ and $\varepsilon > 0$, the set

$$\big\{e \in E^1 \;\big|\; \|x\delta_{e\lambda}\| > \varepsilon, (e,\lambda) \in E^1{}_d{\times_r}\, \Lambda_l\big\}$$

has a compact closure. Thus this is true for all $x \in K(H_l) + \Phi_l\big(\mathcal{K}(C_d(E^1))\big)$.

Take $f \in C_0(E^0)$ with $f(v) \neq 0$ for some $v \in X^0_{\mathrm{sg}}$. Since $\operatorname{Orb}^+(v_0)$ is dense in X^0, we have $X^0_{\mathrm{sce}} = \emptyset$. Hence $v \in X^0_{\mathrm{inf}}$. There exists $\varepsilon > 0$ such that $W = \{w \in E^0 \mid |f(w)| > \varepsilon\}$ is a neighborhood of v. Since $v \in X^0_{\mathrm{inf}}$, the closure of $r^{-1}(W) \cap X^1$ is not compact. Since $\|T_l^0(f)\delta_{e\lambda}\| = |f(r(e\lambda))| = |f(r(e))|$, we have

$$\begin{aligned}
&\big\{e \in E^1 \;\big|\; \|T_l^0(f)\delta_{e\lambda}\| > \varepsilon, (e,\lambda) \in E^1{}_d{\times_r}\, \Lambda_l\big\} \\
&\qquad = \big\{e \in E^1 \;\big|\; |f(r(e))| > \varepsilon, d(e) \in \operatorname{Orb}^+(v_0)\big\} = r^{-1}(W) \cap d^{-1}(\operatorname{Orb}^+(v_0))
\end{aligned}$$

whose closure is not compact. This shows $T_l^0(f) \notin K(H_l) + \Phi_l\big(\mathcal{K}(C_d(E^1))\big)$. Hence we have $Z_{\ker\psi'_l} = X^0_{\mathrm{sg}}$. Therefore we get $\rho_{\ker\psi'_l} = \rho'_{X^0}$.

Since $\ker\psi'_l$ is gauge-invariant by Proposition 12.19, we have $\ker\psi'_l = I_{\rho'_{X^0}}$. □

13. Primitivity of $\mathcal{O}(E)$

A C^*-algebra is said to be *primitive* if 0 is a primitive ideal. Equivalently, a C^*-algebra is primitive if and only if it has a faithful irreducible representation. On the primitivity of $\mathcal{O}(E)$, we have the following.

Proposition 13.1. *For a topological graph E, consider the following three conditions.*

(i) *E is topologically free and $E^0 = \overline{\mathrm{Orb}(v, e)}$ for some $v \in E^0$ and a negative orbit e of v.*
(ii) *the C^*-algebra $\mathcal{O}(E)$ is primitive.*
(iii) *E is topologically free and topologically transitive.*

Then we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii)*. When E^0 is second countable, the three conditions are equivalent.*

Proof. The condition (i) is equivalent to $E^0 \in \mathcal{M}'_{\mathrm{aper}}(E)$ by Proposition 11.3. Hence we have (i)$\Rightarrow$(ii) by Theorem 12.1. Since a primitive C^*-algebra is prime, the implication (ii)$\Rightarrow$(iii) follows from Theorem 10.3. When E^0 is second countable, (iii) implies (i) by Proposition 10.2. $\square$

The converses of the two implications in Proposition 13.1 are not true in general as we will see the following two examples.

Example 13.2 (A topological graph satisfying (ii) but not (i))**.**
Let μ be the Haar measure on $\mathbb{T}$. An irrational rotation α on $\mathbb{T}$ preserves the measure μ. Hence α induces the automorphism $\bar{\alpha}$ on the commutative von Neumann algebra $L^\infty(\mathbb{T}, \mu)$. Let X be the spectrum of $L^\infty(\mathbb{T}, \mu)$ which is considered as a commutative C^*-algebra. Thus X is a compact hyperstonean space such that $C(X) \cong L^\infty(\mathbb{T}, \mu)$. The automorphism $\bar{\alpha}$ of $L^\infty(\mathbb{T}, \mu)$ gives us a homeomorphism σ on X. Thus we get a dynamical system $\Sigma = (X, \sigma)$. Since μ is non-atomic, every orbit of Σ is not dense in X by [T2, Proposition 1.2 (1)]. Hence the topological graph $E_\Sigma = (X, X, \mathrm{id}_X, \sigma)$ does not satisfy (i). We will show that $\mathcal{O}(E_\Sigma)$ is primitive.

Let us consider the covariant representation $\{\pi, u\}$ of $\Sigma = (X, \sigma)$ on $L^2(\mathbb{T}, \mu)$ such that $\pi\colon C(X) \cong L^\infty(\mathbb{T}, \mu) \to B(L^2(\mathbb{T}, \mu))$ is defined by a multiplication, and the unitary $u \in B(L^2(\mathbb{T}, \mu))$ is defined from the irrational rotation α on $\mathbb{T}$. Since the irrational rotation α on $(\mathbb{T}, \mu)$ is free, the dynamical system $\Sigma = (X, \sigma)$ is topologically free by [T2, Proposition 1.2 (3)]. Hence the covariant representation $\{\pi, u\}$ gives a faithful representation $\psi\colon \mathcal{O}(E_\Sigma) \to B(L^2(\mathbb{T}, \mu))$. Since the irrational rotation α is ergodic, ψ is irreducible. Thus the C^*-algebra $\mathcal{O}(E_\Sigma)$ is primitive.

Example 13.3 (A topological graph satisfying (iii) but not (ii))**.**
Let $E = (E^0, E^1, d, r)$ be the discrete graph in Example 4.15. Namely E^0 is the set of all finite subsets of an uncountable set X,

$$E^1 = \big\{(x; v) \bigm| v \in E^0 \text{ and } x \in v\big\},$$

$d((x; v)) = v$ and $r((x; v)) = v \setminus \{x\}$ for $(x; v) \in E^1$. For a positive integer n and $v \in E^0$, let $v^{(n)}$ be the set of n-tuples $x = (x_1, x_2, \ldots, x_n) \in v^n$ such that $x_k \neq x_l$ for $k \neq l$. Note that $v^{(1)}$ is identified with v. For $v \in E^0$, $|v| \in \mathbb{N}$ denotes the number of elements of v. When $|v| < n$ we have $v^{(n)} = \emptyset$, and when $|v| = m \geq n$ we have $|v^{(n)}| = m!/(m-n)!$. For $\big((x_1; v_1), (x_2; v_2), \ldots, (x_n; v_n)\big) \in E^n$, $x = (x_1, x_2, \ldots, x_n)$ is in

$v^{(n)}$ where $v = v_n \in E^0$. Conversely for $v \in E^0$ and $x = (x_1, x_2, \ldots, x_n) \in v^{(n)}$, we have $\big((x_1; v_1), (x_2; v_2), \ldots, (x_n; v_n)\big) \in E^n$ where $v_k = v \setminus \{x_{k+1}, \ldots, x_n\}$ for $k = 1, 2, \ldots, n$. By these correspondences, we will identify E^n with the set

$$\big\{(x; v) \bigm| v \in E^0 \text{ and } x \in v^{(n)}\big\},$$

For $(x; v) \in E^n$ where $x = (x_1, x_2, \ldots, x_n) \in v^{(n)}$, we have $d^n((x; v)) = v$ and $r^n((x; v)) = v \setminus \{x_1, x_2, \ldots, x_n\}$. In order to save the notation, we set $v^{(0)} = \{\emptyset\}$ for $v \in E^0$ and identify E^0 with $\{(\emptyset; v) \mid v \in E^0\}$. For $v \in E^0$, we set $v^{(*)} = \coprod_{n \in \mathbb{N}} v^{(n)} = \coprod_{n=0}^{|v|} v^{(n)}$.

We put $s_{(x;v)} = t^n(\delta_{(x;v)}) \in \mathcal{O}(E)$ for $(x; v) \in E^n$ and $n \in \mathbb{N}$, where $\delta_{(x;v)} \in C_d(E^n)$ is the characteristic function of $\{(x; v)\}$. The linear space

$$\operatorname{span}\big\{s_{(x;v)} s_{(y;v)}^* \bigm| v \in E^0 \text{ and } x, y \in v^{(*)}\big\}$$

is dense in the C^*-algebra $\mathcal{O}(E)$.

Proposition 13.4. *The C^*-algebra $\mathcal{O}(E)$ is prime, but not primitive.*

Proof. We had already seen that E^0 is a maximal head. Hence E is topologically transitive by Proposition 10.2. Since E has no loops, E is topologically free. By Theorem 10.3, the C^*-algebra $\mathcal{O}(E)$ is prime.

Take an irreducible representation $\psi\colon \mathcal{O}(E) \to B(H)$, and we will show that ψ is not faithful. Choose $\xi \in H$ arbitrary. For $n \in \mathbb{N}$, we set $\Omega_n \subset E^0$ by

$$\Omega_n = \big\{v \in E^0 \bigm| \psi(s_{(x;v)} s_{(x;v)}^*)\xi \neq 0 \text{ for some } x \in v^{(n)}\big\}.$$

For each $n \in \mathbb{N}$, Ω_n is countable because $\{s_{(x;v)} s_{(x;v)}^*\}_{v \in E^0, x \in v^{(n)}}$ is an orthogonal family of projections. Therefore $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ is also a countable subset of E^0. Hence we can find $x_0 \in X$ such that $x_0 \notin v$ for all $v \in \Omega$. Let $I \subset \mathcal{O}(E)$ be the closure of

$$\operatorname{span}\big\{s_{(x;v)} s_{(y;v)}^* \bigm| v \in E^0 \text{ with } x_0 \in v, \text{ and } x, y \in v^{(*)}\big\}.$$

By noting that the set $\{v \in E^0 \mid x_0 \in v\}$ is hereditary, we can show that I is an ideal (cf. the proof of Proposition 3.5). Since $x_0 \in v$ implies $v \notin \Omega$, we have $\psi(s_{(y;v)}^*)\xi = 0$ for $v \in E^0$ with $x_0 \in v$. Hence $\psi(a)\xi = 0$ for all $a \in I$. Since ξ is a cyclic vector for the representation ψ, we have $I \subset \ker \psi$. Thus ψ is not injective. □

Remark 13.5. By the proof of Proposition 13.4, we can see that the C^*-algebra $\mathcal{O}(E)$ does not have a faithful cyclic representation. This is the obstacle that N. Weaver used in [We]. We do not know whether this is the only obstacle for prime C^*-algebras to become primitive. Namely the following problem is still open.

Problem 13.6. Is a prime C^*-algebra primitive if it has a faithful cyclic representation?

We give two results on the C^*-algebra $\mathcal{O}(E)$ which suggest that $\mathcal{O}(E)$ is not an "exotic" C^*-algebra.

Proposition 13.7. *The C^*-algebra $\mathcal{O}(E)$ is an inductive limit of finite dimensional C^*-algebras.*

Proof. For $v \in E^0$ we define $A_v \subset \mathcal{O}(E)$ by

$$A_v = \operatorname{span}\{s_{(x;w)} s_{(y;w)}^* \mid w \subset v \text{ and } x, y \in w^{(*)}\}.$$

It is not difficult to see that A_v is a finite dimensional C^*-algebra (see Remark 13.8). It is also easy to see that $v_1 \subset v_2$ implies $A_{v_1} \subset A_{v_2}$ and that $\bigcup_{v \in E^0} A_v$ is dense in $\mathcal{O}(E)$. Thus $\mathcal{O}(E)$ is an inductive limit of finite dimensional C^*-algebras A_v. □

Remark 13.8. For $m \in \mathbb{N}$, we define $a(m) \in \mathbb{N}$ by $a(m) = \sum_{k=0}^{m} m!/k!$. The sequence $\{a(m)\}_{m\in\mathbb{N}}$ is determined by $a(0) = 1$ and $a(m) = ma(m-1) + 1$. For $v \in E^0$, we have $|v^{(*)}| = a(|v|)$. We have

$$A_{v_0} \cong \bigoplus_{v\subset v_0} \mathbb{M}_{a(|v|)} = \bigoplus_{m=0}^{n} \big(\underbrace{\mathbb{M}_{a(m)} \oplus \cdots \oplus \mathbb{M}_{a(m)}}_{\frac{n!}{m!(n-m)!}\text{times}}\big)$$

for $v_0 \in E^0$ where $n = |v_0|$. For each $v \subset v_0$, the matrix units of $\mathbb{M}_{a(|v|)}$ are given by

$$\Big\{ s_{(y;v)}\Big(s_{(\emptyset,v)} - \sum_{x\in v_0\setminus v} s_{(x;v\cup\{x\})}s^*_{(x;v\cup\{x\})}\Big)s^*_{(z;v)}\Big\}_{y,z\in v^{(*)}}.$$

Remark 13.9. For each $v_0 \in E^0$, we can define a Cuntz-Krieger E-family $T_{v_0} = (T^0_{v_0}, T^1_{v_0})$ on A_{v_0} by

$$T^0_{v_0}(\delta_v) = \begin{cases} s_{(\emptyset,v)} & \text{if } v \subset v_0, \\ 0 & \text{otherwise,} \end{cases} \qquad T^1_{v_0}(\delta_{(x;v)}) = \begin{cases} s_{(x;v)} & \text{if } v \subset v_0, \\ 0 & \text{otherwise.} \end{cases}$$

This gives us a $*$-homomorphism $\psi_{v_0}\colon \mathcal{O}(E) \to A_{v_0}$ such that $\psi_{v_0}(x) = x$ for $x \in A_{v_0}$. This proves the next proposition.

Proposition 13.10. *The C^*-algebra $\mathcal{O}(E)$ is residually finite dimensional.*

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Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, 060-0810, JAPAN

E-mail address: katsura@math.sci.hokudai.ac.jp